ON DEGENERATIONS OF SURFACES

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Abstract. This paper surveys and gives a uniform exposition of results contained in [7], [8], [9] and [10]. The subject is degenerations of surfaces, especially to unions of planes. More specifically, we deduce some properties of the smooth surface which is the general fibre of the degeneration from combinatorial features of the central fibre. In particular we show that there are strong constraints on the invariants of a smooth surface which degenerates to configurations of planes.

Finally we consider several examples of embedded degenerations of smooth surfaces to unions of planes.

Our interest in these problems has been raised by a series of interesting articles by Guido Zappa in 1950's.

Contents

1. Introduction

This paper surveys and gives a uniform exposition of results contained in [7], [8], [9] and [10], where we study several properties of flat degenerations of surfaces whose general fibre is a smooth projective algebraic surface and whose central fibre is a reduced, connected surface $X \subset \mathbb{P}^r$, $r \geqslant 3$; a very interesting case is, in particular, when X is assumed to be a *union of planes*.

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As a first application of this approach, we shall see that there are strong constraints on the invariants of a smooth projective surface which degenerates to configurations of planes with global normal crossings or other mild singularities (cf. § 11).

Our results include formulas on the basic invariants of smoothable surfaces (see e.g. Theorems 4.3, 8.1 and 9.9). These formulas are useful in studying a wide range of open problems, as it happens in the curve case, where one considers *stick curves*, i.e. unions of lines with only nodes as singularities. Indeed, as stick curves are used to study moduli spaces of smooth curves and are strictly related to fundamental problems as the *Zeuthen problem* (cf. [31] and [50]), degenerations of surfaces to unions of planes naturally arise in several important instances, like toric geometry (cf. e.g. [4], [24] and [45]) and the study of the behaviour of components of moduli spaces of smooth surfaces and their compactifications. For example, see [38], where the abelian surface case is considered, or several papers related to the $K3$ surface case (see, e.g., [14], [15] and [21]).

Using the techniques developed here (cf. also the original papers [9] and [10]), we are able to extend some results of topological nature of Clemens-Schmid (see Theorem 9.9 and cf. e.g. [43]) and to prove a Miyaoka-Yau type inequality (see Theorem 11.4 and Proposition 11.17).

We expect that degenerations of surfaces to unions of planes will find many applications. For example, applications to secant varieties and the interpolation problem are contained in [13].

It is an open problem to understand when a family of surfaces may degenerate to a union of planes, and in some sense this is one of the most interesting questions in the subject. The techniques we develop here in some cases allow us to conclude that this is not possible. When it is possible, we obtain restrictions on the invariants which may lead to further theorems on classification. The case of scrolls has been treated in [8] (cf. also Theorem 11.16 below). Other possible applications are related, for example, to the problem of bounding the irregularity of surfaces in $\mathbb{P}^4$.

Further applications include the possibility of performing braid monodromy computations (see [16], [41], [42], [52]). We hope that future work will include an analysis of higher-dimensional analogues.

Our interest in degenerations to union of planes has been stimulated by a series of papers by Guido Zappa appeared in the 1940–50's regarding in particular: (1) degenerations of scrolls to unions of planes and (2) the computation of bounds for the topological invariants of an arbitrary smooth projective surface which degenerates to a union of planes (see [55, 56, 57, 58, 59, 60, 61]).

In this paper we shall consider a reduced, connected, projective surface X which is a union of planes — or more generally a union of smooth surfaces — whose singularities are:

- in codimension one, double curves which are smooth and irreducible, along which two surfaces meet transversally;
- multiple points, which are locally analytically isomorphic to the vertex of a cone over a stick curve with arithmetic genus either zero or one and which is projectively normal in the projective space it spans.

These multiple points will be called *Zappatic singularities* and X will be called a *Zappatic surface*. If moreover $X \subset \mathbb{P}^r$, for some positive r, and if all its irreducible components are planes, then X is called a *planar Zappatic surface*.

We will mainly concentrate on the so called *good Zappatic surfaces*, i.e. Zappatic surfaces having only Zappatic singularities whose associated stick curve has one of the following dual graphs (cf. Examples 2.7 and 2.8, Definition 3.5, Figures 3 and 5):

R_n: a chain of length n, with $n \geqslant 3$;
S_n: a fork with $n-1$ teeth, with $n \geqslant 4$;
E_n: a cycle of order n, with $n \geqslant 3$.

Let us call R_n-, S_n-, E_n-*point* the corresponding multiple point of the Zappatic surface X.

We first study some combinatorial properties of a Zappatic surface X (cf. § 3). We then focus on the case in which X is the central fibre of a (an embedded, respectively) flat degeneration $\mathcal{X} \to \Delta$, where Δ is the complex unit disk (and where $\mathcal{X} \subseteq \Delta \times \mathbb{P}^r$, $r \geqslant 3$, is a closed subscheme of relative dimension two, respectively). In this case, we deduce some properties of the general fibre $\mathcal{X}_t$, $t \neq 0$, of the degeneration from the aforementioned properties of the central fibre $\mathcal{X}_0 = X$ (see §'s 5, 8, 9, 10 and 11).

A first instance of this approach can be found in [7], where we gave some partial results on the computation of $h^0(X, \omega_X)$, when X is a Zappatic surface with global normal crossings, i.e. with only E_3-points, and where ω_X is its dualizing sheaf (see Theorem 4.15 in [7]). In the particular case in which X is smoothable, namely if X is the central fibre of a flat degeneration, we recalled that the formula for $h^0(X, \omega_X)$ can be also deduced from the well-known Clemens-Schmid exact sequence (cf. also e.g. [43]).

In this paper we address three main problems.

We define the ω-*genus* of a projective variety Y to be

$$p_\omega(Y) := h^0(Y, \omega_Y), \tag{1.1}$$

where ω_Y is the dualizing sheaf of Y. It is just the *arithmetic* genus, if Y is a reduced curve, and the *geometric* genus, if Y is a smooth surface.

We first extend the computation of the ω-genus to the more general case in which a good Zappatic surface X — considered as a reduced, connected surface on its own — has R_n-, S_n- and E_n-points, for $n \geqslant 3$, as Zappatic singularities (cf. Theorem 4.3). When X is the central fibre of a degeneration $\mathcal{X} \to \Delta$, we then relate the ω-genus of the central fibre to the *geometric genus* of the general one; more precisely, we show that the ω-genus of the fibres of a flat degeneration of surfaces with Zappatic central fibre as above is *constant* (cf. Theorem 9.9 and [10]).

As a second main result, we compute the K^2 of a smooth surface which degenerates to a good Zappatic surface, i.e. we compute $K^2_{\mathcal{X}_t}$, where $\mathcal{X}_t$ is the general fibre of a degeneration $\mathcal{X} \to \Delta$ such that the central fibre $\mathcal{X}_0$ is a good Zappatic surface (see Theorem 8.1 and [9]).

We will then prove a basic inequality, called the *Multiple Point Formula* (cf. Theorem 10.2 and [9]), which can be viewed as a generalization, for good Zappatic singularities, of the well-known Triple Point Formula (see Lemma 10.7 and cf. [20]).

These results follow from a detailed analysis of local properties of the total space $\mathcal{X}$ of the degeneration at a good Zappatic singularity of the central fibre X (cf. §'s 6 and 7).

Furthermore, we apply the computation of K^2 and the Multiple Point Formula to prove several results concerning degenerations of surfaces. Precisely, if χ and g denote, respectively, the Euler-Poincaré characteristic and the sectional genus of the general fibre $\mathcal{X}_t$, for $t \in \Delta \backslash \{0\}$, then (cf. Definition 5.4):

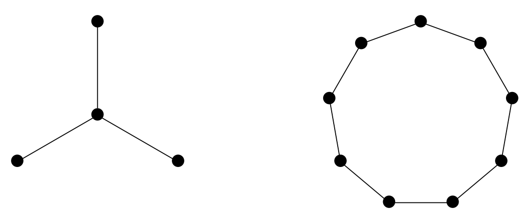
Theorem 1 (cf. Theorem 11.4)**.** *Let $\mathcal{X} \to \Delta$ be a good, planar Zappatic degeneration, where the central fibre $\mathcal{X}_0 = X$ has at most R_3-, E_3-, E_4- and E_5-points. Then*

$$K^2 \leqslant 8\chi + 1 - g. \tag{1.2}$$

Moreover, the equality holds in (1.2) if and only if $\mathcal{X}_t$ is either the Veronese surface in $\mathbb{P}^5$ degenerating to four planes with associated graph S_4 (i.e. with three R_3-points, see Figure 1.a), or an elliptic scroll of degree $n \geqslant 5$ in $\mathbb{P}^{n-1}$ degenerating to n planes with associated graph a cycle E_n (see Figure 1.b).

Furthermore, if $\mathcal{X}_t$ is a surface of general type, then

$$K^2 < 8\chi - g.$$



(a) (b)

FIGURE 1.

In particular, we have:

Corollary (cf. Corollaries 11.10 and 11.12)**.** *Let $\mathcal{X}$ be a good, planar Zappatic degeneration.*

(a) *Assume that $\mathcal{X}_t$, $t \in \Delta \setminus \{0\}$, is a scroll of sectional genus $g \geqslant 2$. Then $\mathcal{X}_0 = X$ has worse singularities than R_3-, E_3-, E_4- and E_5-points.*

(b) *If $\mathcal{X}_t$ is a minimal surface of general type and $\mathcal{X}_0 = X$ has at most R_3-, E_3-, E_4- and E_5-points, then*

$$g \leqslant 6\chi + 5.$$

These improve the main results of Zappa in [60].

Let us describe in more detail the contents of the paper. Section 2 contains some basic results on reducible curves and their dual graphs.

In Section 3, we give the definition of Zappatic singularities and of (planar, good) Zappatic surfaces. We associate to a good Zappatic surface X a graph G_X which encodes the configuration of the irreducible components of X as well as of its Zappatic singularities (see Definition 3.6).

Then we compute from the combinatorial invariants of the associated graph G_X some of the invariants of X, e.g. the Euler-Poincaré characteristic $\chi(\mathcal{O}_X)$, and — when $X \subset \mathbb{P}^r$, $r \geqslant 3$ — the degree $d = \deg(X)$, the sectional genus g, and so on. These computations will be frequently used in later sections, e.g. § 11.

In Section 4 we address the problem of computing the ω-genus of a good Zappatic surface X. Precisely, we compute the cohomology of its structure sheaf, since $p_\omega(X) = h^2(X, \mathcal{O}_X)$, and we prove the following:

Theorem 2. *(cf. Theorem 4.3) Let $X = \bigcup_{i=1}^{v} X_i$ be a good Zappatic surface and let G_X be its associated graph (cf. Definition 3.6). Consider the natural map*

$$\Phi_X : \bigoplus_{i=1}^{v} H^1(X_i, \mathcal{O}_{X_i}) \to \bigoplus_{1 \leqslant i < j \leqslant v} H^1(C_{ij}, \mathcal{O}_{C_{ij}}),$$

where $C_{ij} = X_i \cap X_j$ if X_i and X_j meet along a curve, or $C_{ij} = \emptyset$ otherwise (cf. Definition 4.1). Then:

$$p_\omega(X) = h^2(G_X, \mathbb{C}) + \sum_{i=1}^{v} p_g(X_i) + \dim(\operatorname{coker}(\Phi_X)). \tag{1.3}$$

In particular, we have:

Corollary. *Let X be a good planar Zappatic surface. Then,*

$$p_\omega(X) = b_2(G_X).$$

Remark 1. It is well-known that, for smooth surfaces S, the geometric genus $p_g(S)$ is a topological invariant of S. From Formula (1.3), it follows that also the ω-genus $p_\omega(X)$ is a topological invariant of any good Zappatic surface X.

In order to prove the above results, we exploit the natural injective resolution of the sheaf $\mathcal{O}_X$ in terms of the structure sheaves of the irreducible components of X and of its singular locus. An alternative, and in some sense dual, approach is via the interpretation of the global sections of ω_X as collections of meromorphic 2-forms on the irreducible components of X, having poles along the double curves of X with suitable matching conditions. This interpretation makes it possible, in principle, to compute $h^0(X, \omega_X)$ by computing the number of such independent collections of forms. This is the viewpoint taken in [7], where we discussed only the normal crossings case. However, the approach taken here leads more quickly and neatly to our result.

In Section 5 we give the definition of Zappatic degenerations of surfaces and we recall some properties of smooth surfaces which degenerate to Zappatic ones.

In Section 6 we recall the notions of *minimal singularity* and *quasi-minimal singularity*, which are needed to study the singularities of the total space $\mathcal{X}$ of a degeneration of surfaces at a good Zappatic singularity of its central fibre $\mathcal{X}_0 = X$ (cf. also [34] and [35]).

Section 7 is devoted to studying (partial and total) resolutions of the singularities that the total space $\mathcal{X}$ of a degeneration of surfaces at a good Zappatic singularity of its central fibre.

The local analysis of minimal and quasi-minimal singularities of $\mathcal{X}$ is fundamental in § 8, where we compute $K^2_{\mathcal{X}_t}$, for $t \in \Delta \setminus \{0\}$, when $\mathcal{X}_t$ is the general fibre of a degeneration such that the central fibre is a good Zappatic surface. More precisely, we prove the following main result (see Theorem 8.1 and [9]):

Theorem 3. *Let $\mathcal{X} \to \Delta$ be a degeneration of surfaces whose central fibre is a good Zappatic surface $X = \mathcal{X}_0 = \bigcup_{i=1}^{v} X_i$. Let $C_{ij} := X_i \cap X_j$ be a smooth (possibly reducible) curve of the double locus of X, considered as a curve on X_i, and let g_{ij} be its geometric genus, $1 \leqslant i \neq j \leqslant v$. Let v and e be the number of vertices and edges of the graph G_X associated to X. Let f_n, r_n, s_n be the number of E_n-, R_n-, S_n-points of X, respectively. If $K^2 := K^2_{\mathcal{X}_t}$, for*

$t \neq 0$, then:

$$K^2 = \sum_{i=1}^{v} \left(K_{X_i}^2 + \sum_{j \neq i} (4g_{ij} - C_{ij}^2) \right) - 8e + \sum_{n \geqslant 3} 2nf_n + r_3 + k \tag{1.4}$$

where k depends only on the presence of R_n- and S_n-points, for $n \geqslant 4$, and precisely:

$$\sum_{n \geqslant 4} (n-2)(r_n + s_n) \leqslant k \leqslant \sum_{n \geqslant 4} \left((2n-5)r_n + \binom{n-1}{2} s_n \right). \tag{1.5}$$

In the case that the central fibre is also planar, we have the following:

Corollary (cf. Corollary 8.4)**.** *Let $\mathfrak{X} \to \Delta$ be an embedded degeneration of surfaces whose central fibre is a good, planar Zappatic surface $X = \mathfrak{X}_0 = \bigcup_{i=1}^{v} \Pi_i$. Then:*

$$K^2 = 9v - 10e + \sum_{n \geqslant 3} 2nf_n + r_3 + k \tag{1.6}$$

where k is as in (1.5) *and depends only on the presence of R_n- and S_n-points, for $n \geqslant 4$.*

The inequalities in the theorem and the corollary above reflect deep geometric properties of the degeneration. For example, if $\mathfrak{X} \to \Delta$ is a degeneration with central fibre X a Zappatic surface which is the union of four planes having only a R_4-point, Theorem 3 states that $8 \leqslant K^2 \leqslant 9$. The two values of K^2 correspond to the fact that X, which is the cone over a stick curve C_{R_4} (cf. Example 2.7), can be smoothed either to the Veronese surface, which has $K^2 = 9$, or to a rational normal quartic scroll in $\mathbb{P}^5$, which has $K^2 = 8$ (cf. Remark 8.22). This in turn corresponds to different local structures of the total space of the degeneration at the R_4-point. Moreover, the local deformation space of a R_4-point is reducible.

Section 10 is devoted to the *Multiple Point Formula* (1.7) below (cf. Definition 5.4, see Theorem 10.2 and [9]):

Theorem 4. *Let X be a good Zappatic surface which is the central fibre of a good Zappatic degeneration $\mathfrak{X} \to \Delta$. Let $\gamma = X_1 \cap X_2$ be the intersection of two irreducible components X_1, X_2 of X. Denote by $f_n(\gamma)$ [$r_n(\gamma)$ and $s_n(\gamma)$, respectively] the number of E_n-points [R_n-points and S_n-points, respectively] of X along γ. Denote by d_γ the number of double points of the total space $\mathfrak{X}$ along γ, off the Zappatic singularities of X. Then:*

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) - r_3(\gamma) - \sum_{n \geqslant 4} (r_n(\gamma) + s_n(\gamma) + f_n(\gamma)) \geqslant d_\gamma \geqslant 0. \tag{1.7}$$

In particular, if X is also planar, then:

$$2 + f_3(\gamma) - r_3(\gamma) - \sum_{n \geqslant 4} (r_n(\gamma) + s_n(\gamma) + f_n(\gamma)) \geqslant d_\gamma \geqslant 0. \tag{1.8}$$

Furthermore, if $d_{\mathfrak{X}}$ denotes the total number of double points of $\mathfrak{X}$, off the Zappatic singularities of X, then:

$$2e + 3f_3 - 2r_3 - \sum_{n \geqslant 4} nf_n - \sum_{n \geqslant 4} (n-1)(s_n + r_n) \geqslant d_{\mathfrak{X}} \geqslant 0. \tag{1.9}$$

In § 11 we apply Theorem 3 and 4 above to prove several generalizations of statements given by Zappa. For example we show that worse singularities than normal crossings are needed in order to degenerate as many surfaces as possible to unions of planes.

In Section 9, we apply the result in § 4 on the ω-genus (i.e. Theorem 2 above) to the case of X a smoothable, good Zappatic surface, namely $X = \mathcal{X}_0$ is the central fibre of a flat degeneration $\mathcal{X} \to \Delta$ of surfaces, where Δ is the spectrum of a DVR (or equivalently the complex unit disk) and each fibre $\mathcal{X}_t = \pi^{-1}(t)$, $0 \neq t \in \Delta$, is smooth.

Precisely, with the same hypotheses of Theorem 2, we prove:

Theorem 5. *(cf. Theorem 9.9) Let $\mathcal{X} \to \Delta$ be a flat degeneration of surfaces parametrized by a disk, such that the central fibre $\mathcal{X}_0 = X$ is good Zappatic and each fibre $\mathcal{X}_t$, $t \neq 0$, is smooth. Then, for any $t \neq 0$, one has:*

$$p_g(\mathcal{X}_t) = p_\omega(X). \tag{1.10}$$

In particular, the ω-genus of the fibres of $\mathcal{X} \to \Delta$ is constant.

Remark 2. Recall that, when X has only E_3-points as Zappatic singularities and it is smoothable, with smooth total space $\mathcal{X}$, (i.e. X is the central fibre of a *semistable degeneration*), it is well–known that (1.10) holds. This has been proved via the *Clemens-Schmid exact sequence* approach, which relates the mixed Hodge theory of the central fibre X to that of the general one $\mathcal{X}_t$, $t \in \Delta \setminus \{0\}$, by means of the monodromy of the total space $\mathcal{X}$ (cf. e.g. [43] for details).

We remark that Theorems 2 and 5 above not only show that (1.10) more generally holds for a smoothable good Zappatic surface, i.e. with R_n-, S_n- and E_n-points, $n \geqslant 3$, as Zappatic singularities and with (possibly) singular total space $\mathcal{X}$, but mainly they extend the Clemens-Schmid approach since the computation of $p_\omega(X)$ is independent of the fact that X is the central fibre of a degeneration.

To prove Theorem 5, we use the construction performed in § 7 of a normal crossing reduction $\bar{\pi} : \bar{\mathcal{X}} \to \Delta$ of π, i.e. $\bar{\mathcal{X}} \to \mathcal{X}$ is a resolution of singularities of $\mathcal{X}$ and the support of its central fibre $\bar{\mathcal{X}}_0$ has global normal crossings (cf. Remark 5.2). Then we apply the results in Chapter II of [33] in order to get a semistable reduction $\tilde{\pi} : \tilde{\mathcal{X}} \to \Delta$ of π. This enables us to deduce the topological properties of the fibres of $\tilde{\mathcal{X}}$ from those of X, with the assistance of the Clemens-Schmid exact sequence (cf. e.g. [43]).

In the last section we exhibit several examples of degenerations of smooth surfaces to good Zappatic ones, some of them contained also in [8], in particular with the central fibre having only R_3-, E_n-, $3 \leqslant n \leqslant 6$, points.

We conclude the paper with Appendix A, where we collect several definitions and well-known results concerning connections between commutative homological Algebra and projective Geometry.

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2. Reducible curves and associated graphs

Let C be a projective curve and let C_i, $i = 1, \ldots, n$, be its irreducible components. We will assume that:

- C is connected and reduced;
- C has at most nodes as singularities;

- the curves $C_i, i = 1, \dots, n$, are smooth.

If two components $C_i,\ C_j,\ i < j$, intersect at m_{ij} points, we will denote by $P_{ij}^h,\ h = 1, \dots, m_{ij}$, the corresponding nodes of C.

We can associate to this situation a simple (i.e. with no loops), weighted connected graph G_C, with vertex v_i weighted by the genus g_i of C_i:

- whose vertices $v_1, \dots, v_n$, correspond to the components $C_1, \dots,\ C_n$;
- whose edges $\eta_{ij}^h,\ i < j,\ h = 1, \dots, m_{ij}$, joining the vertices v_i and v_j, correspond to the nodes P_{ij}^h of C.

We will assume the graph to be *lexicographically oriented*, i.e. each edge is assumed to be oriented from the vertex with lower index to the one with higher.

We will use the following notation:

- v is the number of vertices of G_C, i.e. $v = n$;
- e is the number of edges of G_C;
- $\chi(G_C) = v - e$ is the Euler-Poincaré characteristic of G_C;
- $h^1(G_C) = 1 - \chi(G_C)$ is the first Betti number of G_C.

Notice that conversely, given any simple, connected, weighted (oriented) graph G, there is some curve C such that $G = G_C$.

One has the following basic result:

Theorem 2.1. *(cf.* [7, Theorem 2.1]*) In the above situation*

$$\chi(\mathcal{O}_C) = \chi(G_C) - \sum_{i=1}^{v} g_i = v - e - \sum_{i=1}^{v} g_i. \tag{2.2}$$

Proof. Let $\nu : \tilde{C} \to C$ be the normalization morphism; this defines the exact sequence of sheaves on C:

$$0 \to \mathcal{O}_C \to \nu_*(\mathcal{O}_{\tilde{C}}) \to \underline{\tau} \to 0, \tag{2.3}$$

where $\underline{\tau}$ is a skyscraper sheaf supported on $\mathrm{Sing}(C)$. Since the singularities of C are only nodes, one easily determines $H^0(C, \underline{\tau}) \cong \mathbb{C}^e$. Therefore, by the exact sequence (2.3), one gets

$$\chi(\mathcal{O}_C) = \chi(\nu_*(\mathcal{O}_{\tilde{C}})) - e.$$

By the Leray isomorphism and by the fact that ν is finite, one has $\chi(\nu_*(\mathcal{O}_{\tilde{C}})) = \chi(\mathcal{O}_{\tilde{C}})$. Since $\tilde{C}$ is a disjoint union of the $v = n$ irreducible components of C, one has $\chi(\mathcal{O}_{\tilde{C}}) = v - \sum_{i=1}^{v} g_i$, which proves (2.2). (Cf. also [2] for another proof.) □

We remark that Formula (2.2) is equivalent to (cf. Proposition 3.14):

$$p_a(C) = h^1(G_C) + \sum_{i=1}^{v} g_i. \tag{2.4}$$

Notice that C is Gorenstein, i.e. the dualizing sheaf ω_C is invertible. We define the *ω-genus* of C to be

$$p_\omega(C) := h^0(C, \omega_C). \tag{2.5}$$

Observe that, when C is smooth, the ω-genus coincides with the geometric genus of C.

In general, by the Riemann-Roch theorem, one has

$$p_\omega(C) = p_a(C) = h^1(G_C) + \sum_{i=1}^{v} g_i = e - v + 1 + \sum_{i=1}^{v} g_i. \tag{2.6}$$

If we have a flat family $\mathcal{C} \to \Delta$ over a disc Δ with general fibre $\mathcal{C}_t$ smooth and irreducible of genus g and special fibre $\mathcal{C}_0 = C$, then we can combinatorially compute g via the formula:

$$g = p_a(C) = h^1(G_C) + \sum_{i=1}^{v} g_i.$$

Often we will consider C as above embedded in a projective space $\mathbb{P}^r$. In this situation each curve C_i will have a certain degree d_i, so that the graph G_C can be considered as *double weighted*, by attributing to each vertex the pair of weights (g_i, d_i). Moreover one can attribute to the graph a further marking number, i.e. r the embedding dimension of C.

The total degree of C is

$$d = \sum_{i=1}^{v} d_i$$

which is also invariant by flat degeneration.

More often we will consider the case in which each curve C_i is a line. The corresponding curve C is called a *stick curve*. In this case the double weighting is $(0, 1)$ for each vertex, and it will be omitted if no confusion arises.

It should be stressed that it is not true that for any simple, connected, double weighted graph G there is a curve C in a projective space such that $G_C = G$. For example there is no stick curve corresponding to the graph of Figure 2.

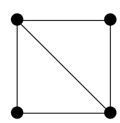


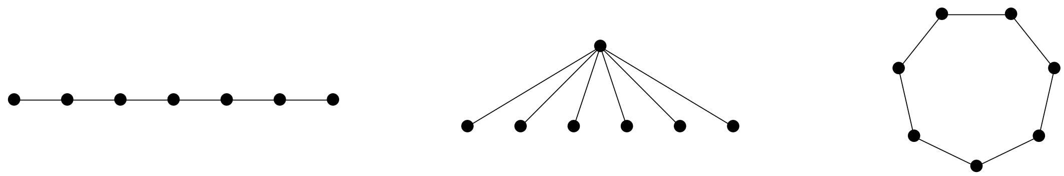
FIGURE 2. Dual graph of an "impossible" stick curve.

We now give two examples of stick curves which will be frequently used in this paper.

Example 2.7. Let T_n be any connected tree with $n \geqslant 3$ vertices. This corresponds to a non-degenerate stick curve of degree n in $\mathbb{P}^n$, which we denote by C_{T_n}. Indeed one can check that, taking a general point p_i on each component of C_{T_n}, the line bundle $\mathcal{O}_{C_{T_n}}(p_1 + \cdots + p_n)$ is very ample. Of course C_{T_n} has arithmetic genus 0 and is a flat limit of rational normal curves in $\mathbb{P}^n$.

We will often consider two particular kinds of trees T_n: a chain R_n of length n and the fork S_n with $n-1$ teeth, i.e. a tree consisting of $n-1$ vertices joining a further vertex (see Figures 3.(a) and (b)). The curve C_{R_n} is the union of n lines $l_1, l_2, \ldots, l_n$ spanning $\mathbb{P}^n$, such that $l_i \cap l_j = \emptyset$ if and only if $1 < |i - j|$. The curve C_{S_n} is the union of n lines $l_1, l_2, \ldots, l_n$ spanning $\mathbb{P}^n$, such that $l_1, \ldots, l_{n-1}$ all intersect l_n at distinct points (see Figure 4).

Example 2.8. Let Z_n be any simple, connected graph with $n \geqslant 3$ vertices and $h^1(Z_n, \mathbb{C}) = 1$. This corresponds to an arithmetically normal stick curve of degree n in $\mathbb{P}^{n-1}$, which we denote



(a) A chain R_n (b) A fork S_n with $n-1$ teeth (c) A cycle E_n

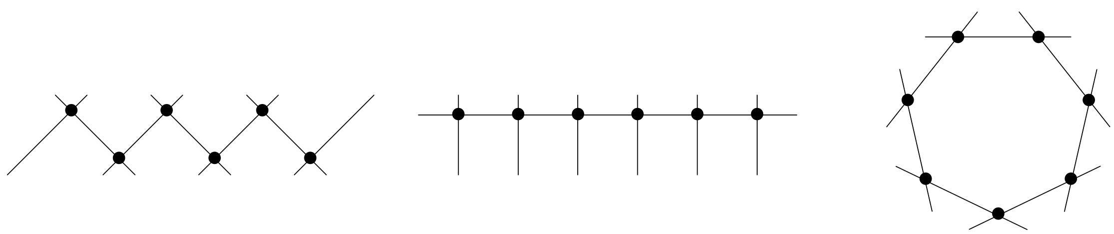
FIGURE 3. Examples of dual graphs.

by C_{Z_n} (as in Example 2.7). The curve C_{Z_n} has arithmetic genus 1 and it is a flat limit of elliptic normal curves in $\mathbb{P}^{n-1}$.

We will often consider the particular case of a cycle E_n of order n (see Figure 3.(c)). The curve C_{E_n} is the union of n lines $l_1, l_2, \ldots, l_n$ spanning $\mathbb{P}^{n-1}$, such that $l_i \cap l_j = \emptyset$ if and only if $1 < |i-j| < n-1$ (see Figure 4).

We remark that C_{E_n} is projectively Gorenstein (i.e. it is projectively Cohen-Macaulay and sub-canonical, cf. Proposition A.50 in Appendix A), because $\omega_{C_{E_n}}$ is trivial, since there is an everywhere non-zero global section of $\omega_{C_{E_n}}$, given by the meromorphic 1-form on each component with residues 1 and -1 at the nodes (in a suitable order).

All the other C_{Z_n}'s, instead, are not Gorenstein because $\omega_{C_{Z_n}}$, although of degree zero, is not trivial. Indeed a graph Z_n, different from E_n, certainly has a vertex with valence 1. This corresponds to a line l such that $\omega_{C_{Z_n}} \otimes \mathcal{O}_l$ is not trivial.



C_{R_n}: a chain of n lines, C_{S_n}: a comb with $n-1$ teeth, C_{E_n}: a cycle of n lines.

FIGURE 4. Examples of stick curves.

3. ZAPPATIC SURFACES AND ASSOCIATED GRAPHS

We will now give a parallel development, for surfaces, to the case of curves recalled in the previous section. Before doing this, we need to introduce the singularities we will allow (cf. [7, § 3]).

Definition 3.1 (Zappatic singularity)**.** Let X be a surface and let $x \in X$ be a point. We will say that x is a *Zappatic singularity* for X if (X, x) is locally analytically isomorphic to a pair (Y, y) where Y is the cone over either a curve C_{T_n} or a curve $C_{Z_n}, n \geqslant 3$, and y is the vertex of the cone. Accordingly we will say that x is either a T_n- or a Z_n*-point* for X.

Observe that either T_n- or Z_n-points are not classified by n, unless $n = 3$.

We will consider the following situation.

Definition 3.2 (Zappatic surface)**.** Let X be a projective surface with its irreducible components $X_1, \ldots, X_v$. We will assume that X has the following properties:

- X is reduced and connected in codimension one;
- $X_1, \dots, X_v$ are smooth;
- the singularities in codimension one of X are at most double curves which are smooth and irreducible along which two surfaces meet transversally;
- the further singularities of X are Zappatic singularities.

A surface like X will be called a *Zappatic surface*. If moreover X is embedded in a projective space $\mathbb{P}^r$ and all of its irreducible components are planes, we will say that X is a *planar Zappatic surface*. In this case, the irreducible components of X will sometimes be denoted by Π_i instead of X_i, $1 \leqslant i \leqslant v$.

Notation 3.3. Let X be a Zappatic surface. Let us denote by:

- X_i: an irreducible component of X, $1 \leqslant i \leqslant v$;
- $C_{ij} := X_i \cap X_j$, $1 \leqslant i \neq j \leqslant v$, if X_i and X_j meet along a curve, otherwise set $C_{ij} = \emptyset$. We assume that each C_{ij} is smooth but not necessarily irreducible;
- g_{ij} : the geometric genus of C_{ij}, $1 \leqslant i \neq j \leqslant v$; i.e. g_{ij} is the sum of the geometric genera of the irreducible (equiv., connected) components of C_{ij};
- $C := \mathrm{Sing}(X) = \cup_{i<j} C_{ij}$: the union of all the double curves of X;
- $\Sigma_{ijk} := X_i \cap X_j \cap X_k$, $1 \leqslant i \neq j \neq k \leqslant v$, if $X_i \cap X_j \cap X_k \neq \emptyset$, otherwise $\Sigma_{ijk} = \emptyset$;
- m_{ijk} : the cardinality of the set Σ_{ijk};
- P^h_{ijk} : the Zappatic singular point belonging to Σ_{ijk}, for $h = 1, \dots, m_{ijk}$.

Furthermore, if $X \subset \mathbb{P}^r$, for some r, we denote by

- $d = \deg(X)$: the degree of X;
- $d_i = \deg(X_i)$: the degree of X_i, $i \leqslant i \leqslant v$;
- $c_{ij} = \deg(C_{ij})$: the degree of C_{ij}, $1 \leqslant i \neq j \leqslant v$;
- D : a general hyperplane section of X;
- g : the arithmetic genus of D;
- D_i : the (smooth) irreducible component of D lying in X_i, which is a general hyperplane section of X_i, $1 \leqslant i \leqslant v$;
- g_i : the genus of D_i, $1 \leqslant i \leqslant v$.

Notice that if X is a planar Zappatic surface, then each C_{ij}, when not empty, is a line and each non-empty set Σ_{ijk} is a singleton.

Remark 3.4. Observe that a Zappatic surface X is Cohen-Macaulay. More precisely, X has global normal crossings except at points T_n, $n \geqslant 3$, and Z_m, $m \geqslant 4$. Thus the dualizing sheaf ω_X is well-defined. If X has only E_n-points as Zappatic singularities, then X is Gorenstein, hence ω_X is an invertible sheaf.

Definition 3.5 (Good Zappatic surface)**.** The *good Zappatic singularities* are the

- R_n-points, for $n \geqslant 3$,
- S_n-points, for $n \geqslant 4$,
- E_n-points, for $n \geqslant 3$,

which are the Zappatic singularities whose associated stick curves are respectively C_{R_n}, C_{S_n}, C_{E_n} (see Examples 2.7 and 2.8, Figures 3, 4 and 5).

A *good Zappatic surface* is a Zappatic surface with only good Zappatic singularities.

To a good Zappatic surface X we can associate an oriented complex G_X, which we will also call the *associated graph* to X.

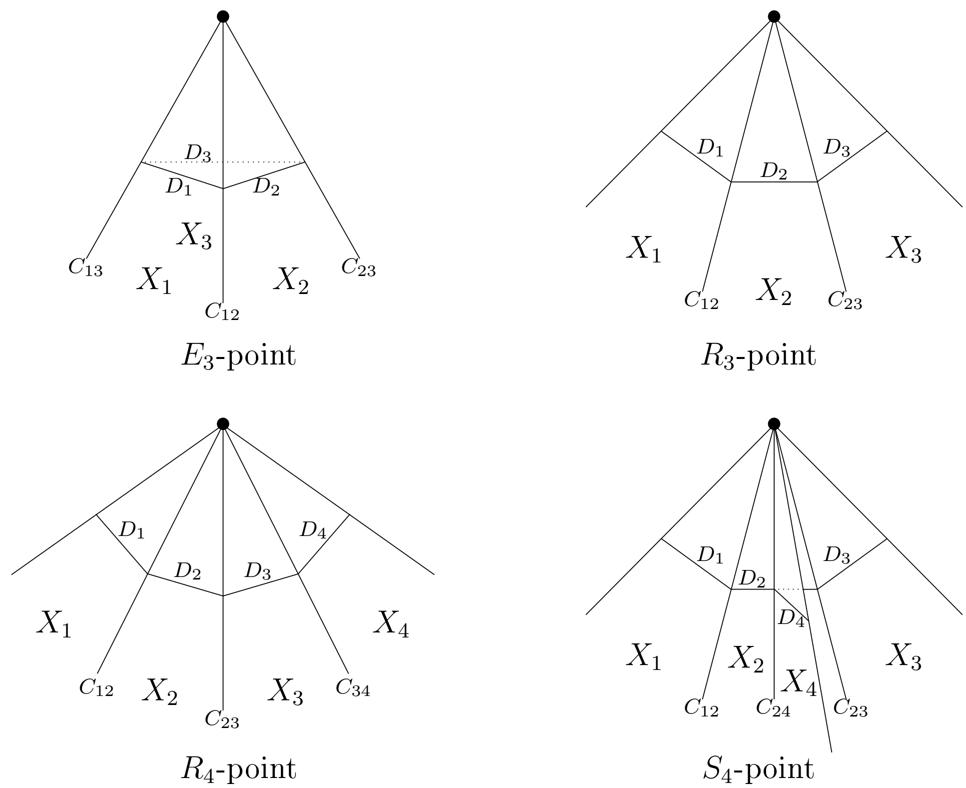


FIGURE 5. Examples of good Zappatic singularities.

Definition 3.6 (The associated graph to X)**.** Let X be a good Zappatic surface with Notation 3.3. The graph G_X associated to X is defined as follows (cf. Figure 6):

- each surface X_i corresponds to a vertex v_i;
- each irreducible component of the double curve $C_{ij} = C_{ij}^1 \cup \cdots \cup C_{ij}^{h_{ij}}$ corresponds to an edge e_{ij}^t, $1 \leqslant t \leqslant h_{ij}$, joining v_i and v_j. The edge $e_{ij}^t, i < j$, is oriented from the vertex v_i to the one v_j. The union of all the edges e_{ij}^t joining v_i and v_j is denoted by $\tilde{e}_{ij}$, which corresponds to the (possibly reducible) double curve C_{ij};
- each E_n-point P of X is a face of the graph whose n edges correspond to the double curves concurring at P. This is called a *n-face* of the graph;
- for each R_n-point P, with $n \geqslant 3$, if $P \in X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_n}$, where X_{i_j} meets X_{i_k} along a curve $C_{i_j i_k}$ only if $1 = |j-k|$, we add in the graph a *dashed edge* joining the vertices corresponding to X_{i_1} and X_{i_n}. The dashed edge e_{i_1,i_n}, together with the other $n-1$ edges $e_{i_j,i_{j+1}}$, $j = 1, \ldots, n-1$, bound an *open n-face* of the graph;
- for each S_n-point P, with $n \geqslant 4$, if $P \in X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_n}$, where $X_{i_1}, \ldots, X_{i_{n-1}}$ all meet X_{i_n} along curves $C_{i_j i_n}$, $j = 1, \ldots, n-1$, concurring at P, we mark this in the graph by an *n-angle* spanned by the edges corresponding to the curves $C_{i_j i_n}$, $j = 1, \ldots, n-1$.

In the sequel, when we speak of *faces* of G_X we always mean closed faces. Of course each vertex v_i is weighted with the relevant invariants of the corresponding surface X_i. We will usually omit these weights if X is planar, i.e. if all the X_i's are planes.

By abusing notation, we will sometimes denote by G_X also the natural CW-complex associated to the graph G_X.

Since each R_n-, S_n-, E_n-point is an element of some set of points Σ_{ijk} (cf. Notation 3.3), we remark that there can be different faces (as well as open faces and angles) of G_X which are incident on the same set of vertices and edges. However this cannot occur if X is planar.

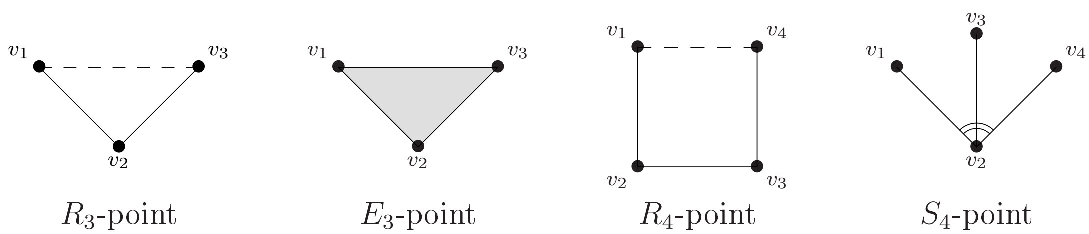


FIGURE 6. Associated graphs of R_3-, E_3-, R_4- and S_4-points (cf. Figure 5).

Consider three vertices v_i, v_j, v_k of G_X in such a way that v_i is joined with v_j and v_k. Assume for simplicity that the double curves C_{ij}, $1 \leqslant i < j \leqslant v$, are irreducible. Then, any point in $C_{ij} \cap C_{ik}$ is either a R_n-, or an S_n-, or an E_n-point, and the curves C_{ij} and C_{ik} intersect transversally, by definition of Zappatic singularities. Hence we can compute the intersection number $C_{ij} \cdot C_{ik}$ by adding the number of closed and open faces and of angles involving the edges e_{ij}, e_{ik}. In particular, if X is planar, for every pair of adjacent edges only one of the following possibilities occur: either they belong to an open face, or to a closed one, or to an angle. Therefore for good, planar Zappatic surfaces we can avoid marking open 3-faces without losing any information (see Figures 6 and 7).

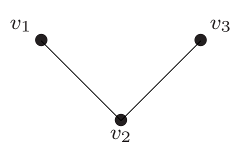


FIGURE 7. Associated graph to a R_3-point in a good, planar Zappatic surface.

As for stick curves, if G is a given graph as above, there does not necessarily exist a good planar Zappatic surface X such that its associated graph is $G = G_X$.

Example 3.7. Consider the graph G of Figure 8. If G were the associated graph to a good planar Zappatic surface X, then X should be a global normal crossing union of 4 planes with 5 double lines and two E_3 points, P_{123} and P_{134}, both lying on the double line C_{13}. Since the lines C_{23} and C_{34} (resp. C_{14} and C_{12}) both lie on the plane X_3 (resp. X_1), they should intersect. This means that the planes X_2, X_4 also should intersect along a line, therefore the edge e_{24} should appear in the graph.

Analogously to Example 3.7, one can easily see that, if the 1-skeleton of G is E_3 or E_4, then in order to have a planar Zappatic surface X such that $G_X = G$, the 2-skeleton of G has to consist of the face bounded by the 1-skeleton.

Let us see two more examples of planar Zappatic surfaces.

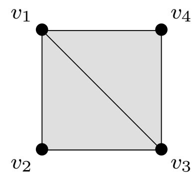


FIGURE 8. Graph associated to an impossible planar Zappatic surface.

Example 3.8. In $\mathbb{P}^4$, with homogeneous coordinates $x_0, \ldots, x_4$, consider the good planar Zappatic surface X which is union of the five planes

$$X_0 = \{x_4 = x_0 = 0\},\ X_i = \{x_i = x_{i-1} = 0\},\ i = 1, 2, 3, 4.$$

The associated graph is a cycle E_5 with no closed faces and the Zappatic singularities are five R_3-points, which, according to the previous remark, we do not mark with open 3-faces.

Example 3.9. In $\mathbb{P}^5$, with affine coordinates $x_1, \ldots, x_5$, the planar Zappatic surface X, which is the union of the five planes

$$X_1 = \{x_4 = x_5 = x_1 = 0\}, \qquad X_2 = \{x_5 = x_1 = x_2 = 0\},$$
$$X_i = \{x_{i-2} = x_{i-1} = x_i = 0\}, \quad i = 3, 4, 5,$$

has an E_5-point at $(0, 0, 0, 0, 0)$. The associated graph is a again a cycle E_5 but with a closed 5-face.

It would be interesting to characterize all the graphs which can be associated to a good Zappatic surface.

Let us see some examples of a good, non-planar, Zappatic surface.

Example 3.10. Consider $X \subset \mathbb{P}^3$ the union of two general quadrics X_1 and X_2 and a general plane X_3. Then, $C_{12} = C_{21}$ is a smooth elliptic quartic in $\mathbb{P}^3$ whereas $C_{13} = C_{31}$ and $C_{23} = C_{32}$ are smooth conics; moreover,

$$X_1 \cap X_2 \cap X_3 = \Sigma_{123} = \Sigma_{213} = \cdots = \Sigma_{321}$$

consists of four distinct points. Hence, G_X has three vertices, three edges (in a cycle) and four triangles (i.e. 3-faces) which are incident on the same set of vertices (equiv. edges).

We can also consider an example of a good Zappatic surface with reducible double curves.

Example 3.11. Consider D_1 and D_2 two general plane curves of degree m and n, respectively. Therefore, they are smooth, irreducible and they transversally intersect each other in mn points. Consider the surfaces:

$$X_1 = D_1 \times \mathbb{P}^1 \ \text{ and } \ X_2 = D_2 \times \mathbb{P}^1.$$

The union of these two surfaces, together with the plane $\mathbb{P}^2 = X_3$ containing the two curves, determines a good Zappatic surface X with only E_3-points as Zappatic singularities.

More precisely, by using Notation 3.3, we have:

- $C_{13} = X_1 \cap X_3 = D_1$, $C_{23} = X_2 \cap X_3 = D_2$, $C_{12} = X_1 \cap X_2 = \sum_{k=1}^{mn} F_k$, where each F_k is a fibre isomorphic to $\mathbb{P}^1$;
- $\Sigma_{123} = X_1 \cap X_2 \cap X_3$ consists of the mn points of the intersection of D_1 and D_2 in X_3.

Observe that C_{12} is smooth but not irreducible. Therefore, the associated graph to X, i.e. G_X, consists of 3 vertices, $mn+2$ edges and mn triangles incident on them.

In order to combinatorially compute some of the invariants of a good Zappatic surface, we need some notation.

Notation 3.12. Let X be a good Zappatic surface (with invariants as in Notation 3.3) and let $G = G_X$ be its associated graph. We denote by

- V : the (indexed) set of vertices of G;
- v : the cardinality of V, i.e. the number of irreducible components of X;
- E : the set of edges of G; this is indexed by the ordered triples $(i,j,t) \in V \times V \times \mathbb{N}$, where $i<j$ and $1 \leqslant t \leqslant h_{ij}$, such that the corresponding surfaces X_i, X_j meet along the curve $C_{ij} = C_{ji} = C_{ij}^1 \cup \cdots \cup C_{ij}^{h_{ij}}$;
- e : the cardinality of E, i.e. the number of irreducible components of double curves in X;
- $\tilde{E}$: the set of double curves C_{ij} of X; this is indexed by the ordered pairs $(i,j) \in V \times V$, where $i<j$, such that the corresponding surfaces X_i, X_j meet along the curve $C_{ij} = C_{ji}$;
- $\tilde{e}$: the cardinality of $\tilde{E}$, i.e. the pairs of vertices of G_X which are joined by at least one edge;
- f_n : the number of n-faces of G, i.e. the number of E_n-points of X, for $n \geqslant 3$;
- $f := \sum_{n \geqslant 3} f_n$, the number of faces of G, i.e. the total number of E_n-points of X, for all $n \geqslant 3$;
- r_n : the number of open n-faces of G, i.e. the number of R_n-points of X, for $n \geqslant 3$;
- $r := \sum_{n \geqslant 3} r_n$, the total number of R_n-points of X, for all $n \geqslant 3$;
- s_n : the number of n-angles of G, i.e. the number of S_n-points of X, for $n \geqslant 4$;
- $s\colon = \sum_{n \geqslant 4} s_n$: the total number of S_n-points of X, for all $n \geqslant 4$;
- $\rho_n\colon = s_n + r_n$, for $n \geqslant 4$, and $\rho_3 = r_3$;
- $\rho\colon = s + r = \sum_{n \geqslant 3} \rho_n$;
- $\tau\colon = \rho + f$, the total number of good Zappatic singularities;
- w_i: the valence of the i^{th} vertex v_i of G, i.e. the number of irreducible double curves lying on X_i;
- $\chi(G) := v - e + f$, i.e. the Euler-Poincaré characteristic of G;
- $G^{(1)}$: the 1-*skeleton* of G, i.e. the graph obtained from G by forgetting all the faces, dashed edges and angles;
- $\chi(G^{(1)}) = v - e$, i.e. the Euler-Poincaré characteristic of $G^{(1)}$.

Remark 3.13. Observe that, when X is a good, planar Zappatic surface, $E = \tilde{E}$ and the 1-skeleton $G_X^{(1)}$ of G_X coincides with the dual graph G_D of the general hyperplane section D of X.

Now we can compute some of the invariants of good Zappatic surfaces.

Proposition 3.14. *(cf.* [7, Proposition 3.12]*) Let $X = \bigcup_{i=1}^{v} X_i \subset \mathbb{P}^r$ be a good Zappatic surface and let $G = G_X$ be its associated graph. Let C be the double locus of X, i.e. the union of the double curves of X, $C_{ij} = C_{ji} = X_i \cap X_j$ and let $c_{ij} = \deg(C_{ij})$. Let D_i be a general hyperplane section of X_i, and denote by g_i its genus. Then the arithmetic genus of a general*

hyperplane section D of X is:

$$g = \sum_{i=1}^{v} g_i + \sum_{1 \leqslant i < j \leqslant v} c_{ij} - v + 1. \tag{3.15}$$

In particular, when X is a good, planar Zappatic surface, then

$$g = e - v + 1 = 1 - \chi(G^{(1)}). \tag{3.16}$$

Proof. Denote by d_i the degree of X_i, $1 \leqslant i \leqslant v$. Then, D is the union of the v irreducible components D_i, $1 \leqslant i \leqslant v$, such that $\deg(D_i) = d_i$ and $d := \deg(D) = \sum_{i=1}^{v} d_i$. Consider its associated graph G_D, defined as in § 2.

Take G, whose indexed set of edges is denoted by E, and consider an edge $e_{ij}^t \in E$ joining its vertices v_i and v_j, $i < j$, which correspond to the irreducible components X_i and X_j, respectively. The edge e_{ij}^t in G corresponds to an irreducible component C_{ij}^t of the double curve C_{ij}, $1 \leqslant t \leqslant h_{ij}$; its degree is denoted by c_{ij}^t, so that $c_{ij} = \sum_{t=1}^{h_{ij}} c_{ij}^t$.

Thus, we have exactly c_{ij} oriented edges in the graph G_D joining its vertices v_i and v_j, which now correspond to the irreducible components of D, D_i and D_j, respectively. These c_{ij} oriented edges correspond to the c_{ij} nodes of the reducible curve $D_i \cup D_j$, which is part of the hyperplane section D.

Now, recall that the Hilbert polynomial of D is, with our notation, $P_D(t) = dt + 1 - g$. On the other hand, $P_D(t)$ equals the number of independent conditions imposed on hypersurfaces $\mathcal{H}$ of degree $t \gg 0$ to contain D.

From what observed above on G_D, it follows that the number of singular points of D is $\sum_{\tilde{e}_{ij} \in \tilde{E}} c_{ij}$. These points impose independent conditions on hypersurfaces $\mathcal{H}$ of degree $t \gg 0$.

Since $t \gg 0$ by assumption, we get that the map

$$H^0(\mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(\mathcal{O}_{D_i}(t))$$

is surjective and that the line bundle $\mathcal{O}_{D_i}(t)$ is non-special on D_i, for each $1 \leqslant i \leqslant v$. Thus, in order for $\mathcal{H}$ to contain D_i we have to impose $d_i t - g_i + 1 - \sum_{j \text{ s.t. } \tilde{e}_{ij} \in \tilde{E}} c_{ij}$ conditions. Therefore the total number of conditions for $\mathcal{H}$ to contain D is:

$$\sum_{\tilde{e}_{ij} \in \tilde{E}} c_{ij} + \sum_{i=1}^{v} \Big(d_i t - g_i + 1 - \sum_{j, \tilde{e}_{ij} \in \tilde{E}} c_{ij} \Big) = \sum_{\tilde{e}_{ij} \in \tilde{E}} c_{ij} + dt - \sum_{i=1}^{v} g_i + v - \sum_{i=1}^{v} \sum_{j, \tilde{e}_{ij} \in \tilde{E}} c_{ij} =$$

$$= dt + v - \sum_{i=1}^{v} g_i - \sum_{\tilde{e}_{ij} \in \tilde{E}} c_{ij},$$

since $\sum_{i=1}^{v} \sum_{j, \tilde{e}_{ij} \in \tilde{E}} c_{ij} = 2 \sum_{\tilde{e}_{ij} \in \tilde{E}} c_{ij}$. This proves (3.15) (cf. Formula (2.6)).

The second part of the statement directly follows from the above computations and from the fact that, in the good planar Zappatic case $g_i = 0$ and $c_{ij} = 1$, for each $i < j$, i.e. G_D coincides with $G^{(1)}$ (cf. Remark 3.13). $\square$

By recalling Notation 3.12, one also has:

Proposition 3.17. *(cf.* [7, Proposition 3.15]*) Let $X = \bigcup_{i=1}^{v} X_i$ be a good Zappatic surface and G_X be its associated graph, whose number of faces is f. Let C be the double locus of X,*

which is the union of the curves $C_{ij} = X_i \cap X_j$. Then:

$$\chi(\mathcal{O}_X) = \sum_{i=1}^{v} \chi(\mathcal{O}_{X_i}) - \sum_{1 \leqslant i<j \leqslant v} \chi(\mathcal{O}_{C_{ij}}) + f. \tag{3.18}$$

In particular, when X is a good, planar Zappatic surface, then

$$\chi(\mathcal{O}_X) = \chi(G_X) = v - e + f. \tag{3.19}$$

Proof. We can consider the sheaf morphism:

$$\bigoplus_{i=1}^{v} \mathcal{O}_{X_i} \xrightarrow{\lambda} \bigoplus_{1 \leqslant i<j \leqslant v} \mathcal{O}_{C_{ij}}, \tag{3.20}$$

defined in the following way: if

$$\pi_{ij} : \bigoplus_{1 \leqslant i<j \leqslant v} \mathcal{O}_{C_{ij}} \to \mathcal{O}_{C_{ij}}$$

denotes the projection on the $(ij)^{\text{th}}$-summand, then

$$(\pi_{ij} \circ \lambda)(h_1, \ldots, h_v) := h_i - h_j.$$

Notice that the definition of λ is consistent with the lexicographic order of the indices and with the lexicographic orientation of the edges of the graph G_X.

Observe that, if $\tilde{X}$ denotes the minimal desingularization of X, then $\tilde{X}$ is isomorphic to the disjoint union of the smooth, irreducible components X_i, $1 \leqslant i \leqslant v$, of X. Therefore, by the very definition of $\mathcal{O}_X$, we see that

$$\ker(\lambda) \cong \mathcal{O}_X.$$

We claim that the morphism λ is not surjective and that its cokernel is a sky-scraper sheaf supported at the E_n-points of X, for $n \geqslant 3$. To show this, we focus on any irreducible component of $C = \bigcup_{1 \leqslant i<j \leqslant v} C_{ij}$, the double locus of X. For simplicity, we shall assume that each curve C_{ij} is irreducible; one can easily extend the same computations to the general case.

Fix any index pair (i, j), with $i < j$, and consider the generator

$$(0, \ldots, 0, 1, 0, \ldots, 0) \in \bigoplus_{1 \leqslant l<m \leqslant v} \mathcal{O}_{C_{lm}}, \tag{3.21}$$

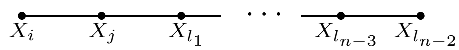
where $1 \in \mathcal{O}_{C_{ij}}$, the $(ij)^{\text{th}}$-summand. The obstructions to lift up this element to an element of $\bigoplus_{1 \leqslant t \leqslant v} \mathcal{O}_{X_t}$ are given by the presence of good Zappatic singularities of X along C_{ij}.

For what concerns the irreducible components of X which are not involved in the intersection determining a good Zappatic singularity on C_{ij}, the element in (3.21) trivially lifts-up to 0 on each of them. Thus, in the sequel, we shall focus only on the irreducible components involved in the Zappatic singularity, which will be denoted by $X_i,\ X_j,\ X_{l_t}$, for $1 \leqslant t \leqslant n-2$.

We have to consider different cases, according to the good Zappatic singularity type lying on the curve $C_{ij} = X_i \cap X_j$.

- Suppose that C_{ij} passes through a R_n-point P of X, for some n; we have two different possibilities. Indeed:

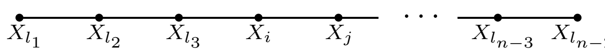
(a) let X_i be an "external" surface for P — i.e. X_i corresponds to a vertex of the associated graph to P which has valence 1. Therefore, we have:



In this situation, the element in (3.21) lifts up to

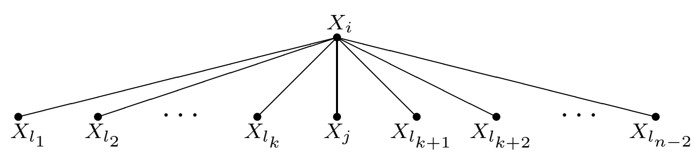
$$(1,0,\dots,0) \in \mathcal{O}_{X_i} \oplus \mathcal{O}_{X_j} \oplus \bigoplus_{1\leqslant t\leqslant n-2} \mathcal{O}_{X_{l_t}}.$$

(b) let X_i be an "internal" surface for P — i.e. X_i corresponds to a vertex of the associated graph to P which has valence 2. Thus, we have a picture like:

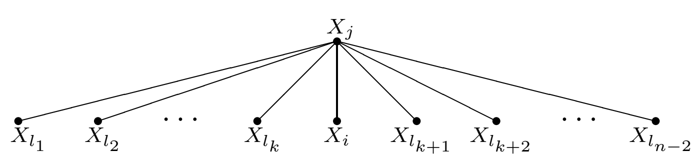


In this case, the element in (3.21) lifts up to the n-tuple having components:
$1 \in \mathcal{O}_{X_i}$,
$0 \in \mathcal{O}_{X_j}$,
$1 \in \mathcal{O}_{X_{l_t}}$, for those X_{l_t}'s corresponding to vertices in the graph associated to P which are on the left of X_i and,
$0 \in \mathcal{O}_{X_{l_k}}$ for those X_{l_k}'s corresponding to vertices in the graph associated to P which are on the right of X_j.

- Suppose that C_{ij} passes through a S_n-point P of X, for any n; as before, we have two different possibilities. Indeed:
(a) let X_i correspond to the vertex of valence $n-1$ in the associated graph to P, i.e.



In this situation, the element in (3.21) lifts up to the n-tuple having components:
$1 \in \mathcal{O}_{X_i}$,
$0 \in \mathcal{O}_{X_j}$,
$1 \in \mathcal{O}_{X_{l_t}}$, for all $1 \leqslant t \leqslant n-2$.
(b) let X_i correspond to a vertex of valence 1 in the associated graph to P. Since $C_{ij} \neq \emptyset$ by assumption, then X_j has to be the vertex of valence $n-1$, i.e. we have the following picture:



Thus, the element in (3.21) lifts up to the n-tuple having components
$1 \in \mathcal{O}_{X_i}$,
$0 \in \mathcal{O}_{X_j}$,
$0 \in \mathcal{O}_{X_{l_t}}$, for all $1 \leqslant t \leqslant n-2$.

- Suppose that C_{ij} passes through an E_n-point P for X. Then, each vertex of the associated graph to P has valence 2. Since such a graph is a cycle, it is clear that no lifting of (3.21) can be done.

To sum up, we see that $\operatorname{coker}(\lambda)$ is supported at the E_n-points of X. Furthermore, if we consider

$$\bigoplus_{1\leqslant i<j\leqslant v} \mathcal{O}_{C_{ij}} \xrightarrow{ev_P} \mathcal{O}_P = \underline{\mathbb{C}}_P, \qquad \bigoplus f_{ij} \mapsto \sum f_{ij}(P)$$

it is clear that, if P is an E_n-point then

$$ev_P\left(\bigoplus_{1\leqslant i<j\leqslant v} \mathcal{O}_{C_{ij}}/\operatorname{Im}(\lambda)\right) \cong \underline{\mathbb{C}}_P.$$

This means that

$$\operatorname{coker}(\lambda) \cong \underline{\mathbb{C}}^f.$$

By the exact sequences

$$0 \to \mathcal{O}_X \to \bigoplus_{1\leqslant i\leqslant v} \mathcal{O}_{X_i} \to \operatorname{Im}(\lambda) \to 0, \qquad 0 \to \operatorname{Im}(\lambda) \to \bigoplus_{1\leqslant i<j\leqslant v} \mathcal{O}_{C_{ij}} \to \underline{\mathbb{C}}^f \to 0,$$

we get (3.18). $\square$

Not all of the invariants of X can be directly computed by the graph G_X. For example, if ω_X denotes the dualizing sheaf of X, the computation of the ω-genus $h^0(X,\omega_X)$, which plays a fundamental role in degeneration theory, is actually much more involved, even if X has mild Zappatic singularities, as we shall see in the next section (cf. also [7]).

To conclude this section, we observe that in the particular case of good, planar Zappatic surface one can determine a simple relation among the numbers of Zappatic singularities, as the next lemma shows.

Lemma 3.22. *(cf.* [9, Lemma 3.16]*) Let G be the associated graph to a good, planar Zappatic surface $X = \bigcup_{i=1}^{v} X_i$. Then, with Notation* (3.12)*, we have*

$$\sum_{i=1}^{v} \frac{w_i(w_i-1)}{2} = \sum_{n\geqslant 3} (nf_n + (n-2)r_n) + \sum_{n\geqslant 4} \binom{n-1}{2} s_n. \tag{3.23}$$

Proof. The associated graph to three planes which form a R_3-point consists of two adjacent edges (cf. Figure 7). The total number of two adjacent edges in G is the left hand side member of (3.23) by definition of valence w_i. On the other hand, an n-face (resp. an open n-face, resp. an n-angle) clearly contains exactly n (resp. $n-2$, resp. $\binom{n-1}{2}$) pairs of adjacent edges. $\square$

4. The ω-genus of a Zappatic surface

The aim of this section is to compute the ω-genus of a good Zappatic surface X, as defined in Formula (1.1). What we will actually do will be to compute the cohomology of the structure sheaf $\mathcal{O}_X$, which is sufficient, since $p_\omega(X) = h^2(X, \mathcal{O}_X)$.

We first define the map Φ_X which appears in the statement of Theorem 2.

Definition 4.1. Let $X = \bigcup_{i=1}^{v} X_i$ be a good Zappatic surface. Let $r_{ij} : H^1(X_i, \mathcal{O}_{X_i}) \to H^1(C_{ij}, \mathcal{O}_{C_{ij}})$ be the restriction map to C_{ij} as a divisor in X_i. We define the natural map:

$$\Phi_X : \bigoplus_{i=1}^{v} H^1(X_i, \mathcal{O}_{X_i}) \to \bigoplus_{1\leqslant i<j\leqslant v} H^1(C_{ij}, \mathcal{O}_{C_{ij}}), \quad \Phi_X(a_i) = -\sum_{j=1}^{i-1} r_{ij}(a_i) + \sum_{j=i+1}^{v} r_{ij}(a_i) \tag{4.2}$$

if $a_i \in H^1(X_i, \mathcal{O}_{X_i})$ and extend Φ_X linearly. When X is clear from the context, we will write simply Φ instead of Φ_X.

The main result of this section is the following (cf. [10, Theorem 3.1] together with the beginning of § 2 and Definition 2.4 in [10]):

Theorem 4.3. *Let $X = \bigcup_{i=1}^v X_i$ be a good Zappatic surface. Then:*

$$p_\omega(X) = h^2(X, \mathcal{O}_X) = h^2(G_X, \mathbb{C}) + \sum_{i=1}^{v} p_g(X_i) + \dim(\mathrm{coker}(\Phi)), \tag{4.4}$$

and

$$h^1(X, \omega_X) = h^1(X, \mathcal{O}_X) = h^1(G_X, \mathbb{C}) + \dim(\ker(\Phi)) \tag{4.5}$$

where G_X is the associated graph to X and $\Phi = \Phi_X$ is the map of Definition 4.1.

Proof. Let $p_1, ..., p_f$ be the E_n–points of X, $n \geqslant 3$. As in the proof of Proposition 3.17 (cf. also Proposition 3.15 in [7]), one has the exact sequence:

$$0 \to \mathcal{O}_X \to \bigoplus_{i=1}^{v} \mathcal{O}_{X_i} \xrightarrow{d_G^0} \bigoplus_{1 \leqslant i < j \leqslant v} \mathcal{O}_{C_{ij}} \xrightarrow{d_G^1} \bigoplus_{h=1}^{f} \mathcal{O}_{p_h} \to 0 \tag{4.6}$$

where the maps are as follows:

- $\mathcal{O}_X \to \bigoplus_{i=1}^v \mathcal{O}_{X_i}$ is the direct sum of the natural restriction maps.
- recall that $d_G^0 : \bigoplus_{i=1}^v \mathcal{O}_{X_i} \to \bigoplus_{1 \leqslant i<j \leqslant v} \mathcal{O}_{C_{ij}}$, can be described by considering the composition of its restriction to each summand $\mathcal{O}_{X_i}$ with the projection to any summand $\mathcal{O}_{C_{hk}}$, with $h < k$. This map sends $g \in \mathcal{O}_{X_i}$ to:
  1. $0 \in \mathcal{O}_{C_{hk}}$, if both h, k are different from i;
  2. $g_{|C_{ik}} \in \mathcal{O}_{C_{ik}}$ if $k > i$;
  3. $-g_{|C_{ki}} \in \mathcal{O}_{C_{ki}}$ if $k < i$;
- the map $d_G^1 : \bigoplus_{1 \leqslant i<j \leqslant v} \mathcal{O}_{C_{ij}} \to \bigoplus_{h=1}^f \mathcal{O}_{p_h}$ again can be described by considering the composition of its restriction to each summand $\mathcal{O}_{C_{ij}}$, with $i < j$, with the projection to any summand $\mathcal{O}_{p_h}$. Suppose p_h is an E_n–point corresponding to a face F_h of G_X such that $\partial F_h = \sum_{1 \leqslant i<j \leqslant v} e_{ij} C_{ij}$, where either $e_{ij} = 0$ or $e_{ij} = \pm 1$. Then this map sends $g \in \mathcal{O}_{C_{ij}}$ to $e_{ij} g(p_h)$.

We note that the induced maps on global sections in each case are the corresponding cochain map for the graph G_X; this motivates the notation for these maps used in (4.6).

Let Λ be the kernel of the sheaf map d_G^1, so that we have two short exact sequences

$$0 \to \mathcal{O}_X \to \bigoplus_{i=1}^{v} \mathcal{O}_{X_i} \to \Lambda \to 0 \tag{4.7}$$

and

$$0 \to \Lambda \to \bigoplus_{1 \leqslant i<j \leqslant v} \mathcal{O}_{C_{ij}} \xrightarrow{d_G^1} \bigoplus_{h=1}^{f} \mathcal{O}_{p_h} \to 0. \tag{4.8}$$

The latter gives the long exact sequence:

$$0 \to H^0(\Lambda) \to \bigoplus_{1\leqslant i<j\leqslant v} H^0(\mathcal{O}_{C_{ij}}) \xrightarrow{d^1_G} \bigoplus_{h=1}^{f} H^0(\mathcal{O}_{p_h}) \to$$

$$\to H^1(\Lambda) \xrightarrow{\beta} \bigoplus_{1\leqslant i<j\leqslant v} H^1(\mathcal{O}_{C_{ij}}) \to 0$$

and since the cokernel of the map d^1_G is $H^2(G_X, \mathbb{C})$, we derive the short exact sequence

$$0 \to H^2(G_X, \mathbb{C}) \to H^1(\Lambda) \xrightarrow{\beta} \bigoplus_{1\leqslant i<j\leqslant v} H^1(\mathcal{O}_{C_{ij}}) \to 0. \tag{4.9}$$

From the short exact sequence (4.7) we have the long exact sequence:

$$0 \to H^0(\mathcal{O}_X) \to \bigoplus_{i=1}^{v} H^0(\mathcal{O}_{X_i}) \to H^0(\Lambda) \to$$

$$\to H^1(\mathcal{O}_X) \to \bigoplus_{i=1}^{v} H^1(\mathcal{O}_{X_i}) \xrightarrow{\alpha} H^1(\Lambda) \to H^2(\mathcal{O}_X) \to \bigoplus_{i=1}^{v} H^2(\mathcal{O}_{X_i}) \to 0.$$

Now $H^1(G_X, \mathbb{C})$ is the kernel of d^1_G (which is $H^0(\Lambda)$) modulo the image of d^0_G, which is the image of the map $\bigoplus_{i=1}^{v} H^0(\mathcal{O}_{X_i}) \to H^0(\Lambda)$ in the first line above. Hence we recognize $H^1(G_X, \mathbb{C})$ as the cokernel of this map, and therefore the second line of the above sequence becomes

$$0 \to H^1(G_X, \mathbb{C}) \to H^1(\mathcal{O}_X) \to \bigoplus_{i=1}^{v} H^1(\mathcal{O}_{X_i}) \xrightarrow{\alpha} H^1(\Lambda) \to H^2(\mathcal{O}_X) \to \bigoplus_{i=1}^{v} H^2(\mathcal{O}_{X_i}) \to 0.$$

Now the composition of the map β with the map α is exactly the map Φ: $\Phi = \beta \circ \alpha$. We claim that α and Φ have the same kernel, which by (4.9) is equivalent to having $\mathrm{Im}(\alpha) \cap H^2(G_X, \mathbb{C}) (= \ker(\beta)) = \{0\}$.

If we are able to show this, then the leftmost part of the above sequence would split off as

$$0 \to H^1(G_X, \mathbb{C}) \to H^1(\mathcal{O}_X) \to \ker(\alpha) = \ker(\Phi) \to 0$$

which would prove the H^1 statement of the theorem. In addition, if this is true, then the natural surjection from the cokernel of α to the cokernel of Φ would have $\ker(\beta) = H^2(G_X, \mathbb{C})$ as its kernel, and we would have $\dim(\mathrm{coker}(\alpha)) = \dim H^2(G_X, \mathbb{C}) + \dim(\mathrm{coker}(\Phi))$. Since the rightmost part of the long exact sequence above splits off as

$$0 \to \mathrm{coker}(\alpha) \to H^2(\mathcal{O}_X) \to \bigoplus_{i=1}^{v} H^2(\mathcal{O}_{X_i}) \to 0$$

we see that the H^2 statement of the theorem follows also.

To prove that $\mathrm{Im}(\alpha) \cap H^2(G_X, \mathbb{C}) = \{0\}$, notice that the sheaf map d^0_G (which has Λ as its image) factors through obvious maps:

$$\bigoplus_{i=1}^{v} \mathcal{O}_{X_i} \to \bigoplus_{i=1}^{v} \mathcal{O}_{C_i} \to \bigoplus_{1\leqslant i<j\leqslant v} \mathcal{O}_{C_{ij}}$$

and therefore the map α on the H^1 level factors as:

$$\bigoplus_{i=1}^{v} H^1(\mathcal{O}_{X_i}) \to \bigoplus_{i=1}^{v} H^1(\mathcal{O}_{C_i}) \to H^1(\Lambda).$$

Moreover one has the short exact sequence:

$$0 \to \mathcal{O}_C \to \bigoplus_{i=1}^{v} \mathcal{O}_{C_i} \to \Lambda \to 0$$

where C is the singular locus of X, and thus we have an exact sequence:

$$H^1(C, \mathcal{O}_C) \to \bigoplus_{i=1}^{v} H^1(\mathcal{O}_{C_i}) \to H^1(\Lambda) \to 0. \tag{4.10}$$

We remark that $H^1(C_i, \mathcal{O}_{C_i})$ [resp. $H^1(C, \mathcal{O}_C)$] is the tangent space at the origin to $\mathrm{Pic}^0(C_i)$ [resp. to $\mathrm{Pic}^0(C)$] which is a $(\mathbb{C}^*)^{\delta_i}$–extension [resp. a $(\mathbb{C}^*)^{\delta}$–extension] of $\bigoplus_{j=1}^{v} \mathrm{Pic}^0(C_{ij})$ [resp. of $\bigoplus_{1 \leqslant i < j \leqslant v} \mathrm{Pic}^0(C_{ij})$], where δ_i [resp. δ] depends on the singular points of C_i [resp. of C].

There are natural restriction maps:

$$a : \bigoplus_{i=1}^{v} \mathrm{Pic}^0(X_i) \to \bigoplus_{i=1}^{v} \mathrm{Pic}^0(C_i)$$

and

$$b : \mathrm{Pic}^0(C) \to \bigoplus_{i=1}^{v} \mathrm{Pic}^0(C_i)$$

which are maps of $\mathbb{C}^*$–extensions of abelian varieties; their differentials at the origin are

$$\bigoplus_{i=1}^{v} H^1(\mathcal{O}_{X_i}) \to \bigoplus_{i=1}^{v} H^1(\mathcal{O}_{C_i})$$

and

$$H^1(C, \mathcal{O}_C) \to \bigoplus_{i=1}^{v} H^1(\mathcal{O}_{C_i})$$

respectively; the latter is the leftmost map of the sequence (4.10).

The map b appears in the following exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & (\mathbb{C}^*)^{\delta} \arrow[r] \arrow[d] & \mathrm{Pic}^0(C) \arrow[r] \arrow[d, "b"] & \bigoplus_{i<j} \mathrm{Pic}^0(C_{ij}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_i (\mathbb{C}^*)^{\delta_i} \arrow[r] & \bigoplus_i \mathrm{Pic}^0(C_i) \arrow[r] & \bigoplus_{i,j} \mathrm{Pic}^0(C_{ij}) \arrow[r] & 0
\end{tikzcd}$$

The vertical map on the right is an injection; indeed, it is the direct sum of diagonal maps $\mathrm{Pic}^0(C_{ij}) \to \mathrm{Pic}^0(C_{ij}) \bigoplus \mathrm{Pic}^0(C_{ji})$. Therefore, if we denote by V the cokernel of the central map b, we have a short exact sequence of cokernels

$$0 \to (\mathbb{C}^*)^{\gamma} \to V \to \bigoplus_{i<j} \mathrm{Pic}^0(C_{ij}) \to 0$$

for some γ; in particular, V is again a $\mathbb{C}^*$-extension of abelian varieties. We now recognize by (4.10) that $H^1(X, \Lambda)$ is the tangent space at the origin to V; moreover the sequence (4.9)

is the map on tangent spaces for the above sequence of groups. In particular the map β is the tangent space map for the projection $V \to \bigoplus_{i<j} \mathrm{Pic}^0(C_{ij})$.

Composing a with the projection of $\bigoplus_{i=1}^{v} \mathrm{Pic}^0(C_i)$ to V gives a map

$$c : \bigoplus_{i=1}^{v} \mathrm{Pic}^0(X_i) \to V$$

whose differential at the origin is the previously encountered map

$$\alpha : \bigoplus_{i=1}^{v} H^1(\mathcal{O}_{X_i}) \to H^1(X, \Lambda).$$

Now $\bigoplus_{i=1}^{v} \mathrm{Pic}^0(X_i)$ is compact, and therefore the image of c in V has finite intersection with the kernel of the projection $V \to \bigoplus_{1 \leqslant i<j \leqslant v} \mathrm{Pic}^0(C_{ij})$. At the tangent space level, this means that the image of α has trivial intersection with the kernel of the map β, which we have identified as $H^2(G_X, \mathbb{C})$, which was to be proved. □

Remark 4.11. Note that, in particular, Formulas (4.4) and (4.5) agree with, and imply, Formula (3.18) that we proved in Proposition 3.17.

In case X is a planar Zappatic surface, Theorem 4.3 implies the following:

Corollary 4.12. *Let X be a good, planar Zappatic surface. Then,*

$$p_\omega(X) = b_2(G_X), \tag{4.13}$$

$$q(X) = b_1(G_X). \tag{4.14}$$

5. Degenerations to Zappatic surfaces

In this section we will focus on flat degenerations of smooth surfaces to Zappatic ones.

Definition 5.1. Let Δ be the spectrum of a DVR (equiv. the complex unit disk). A *degeneration* of relative dimension n is a proper and flat morphism

$$\begin{array}{c} \mathcal{X} \\ \downarrow{\scriptstyle \pi} \\ \Delta \end{array}$$

such that $\mathcal{X}_t = \pi^{-1}(t)$ is a smooth, irreducible, n-dimensional, projective variety, for $t \neq 0$.

If Y is a smooth, projective variety, the degeneration

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] & \subseteq & \Delta \times Y \arrow[dll, "\mathrm{pr}_1"] \\ \Delta & & \end{tikzcd}$$

is said to be an *embedded degeneration* in Y of relative dimension n. When it is clear from the context, we will omit the term *embedded*.

We will say that $\mathcal{X} \to \Delta$ is a *normal crossing degeneration* if the total space $\mathcal{X}$ is smooth and the support X_{red} of the central fibre $X = \mathcal{X}_0$ is a divisor in $\mathcal{X}$ with global normal crossings, i.e. X_{red} is a good Zappatic surface with only E_3-points as Zappatic singularities.

A normal crossing degeneration is called *semistable* (see, e.g., [43]) if the central fibre is reduced.

Remark 5.2. Given a degeneration $\pi : \mathcal{X} \to \Delta$, Hironaka's Theorem on the resolution of singularities implies that there exists a birational morphism $\bar{\mathcal{X}} \to \mathcal{X}$ such that $\bar{\mathcal{X}} \to \Delta$ is a normal crossing degeneration, which we will call a *normal crossing reduction* of π.

Given a degeneration $\pi : \mathcal{X} \to \Delta$, the Semistable Reduction Theorem (see Theorem on p. 53–54 in [33]) states that there exists a base change $\beta : \Delta \to \Delta$, defined by $\beta(t) = t^m$, for some m, a semistable degeneration $\tilde{\pi} : \tilde{\mathcal{X}} \to \Delta$ and a diagram

$$\begin{tikzcd} \tilde{\mathcal{X}} \arrow[r, "\psi"] \arrow[dr, "\tilde{\pi}"'] & \mathcal{X}_\beta \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ & \Delta \arrow[r, "\beta"] & \Delta \end{tikzcd} \tag{5.3}$$

such that the square is Cartesian and $\psi : \tilde{\mathcal{X}} \to \mathcal{X}_\beta$ is a birational morphism obtained by blowing-up a suitable sheaf of ideals on $\mathcal{X}_\beta$. This is called a *semistable reduction* of π.

From now on, we will be concerned with degenerations of relative dimension two, namely degenerations of smooth, projective surfaces.

Definition 5.4. (cf. [9, Definition 4.2]) Let $\mathcal{X} \to \Delta$ be a degeneration (equiv. an embedded degeneration) of surfaces. Denote by $\mathcal{X}_t$ the general fibre, which is by definition a smooth, irreducible and projective surface; let $X = \mathcal{X}_0$ denote the central fibre. We will say that the degeneration is *Zappatic* if X is a Zappatic surface, the total space $\mathcal{X}$ is smooth except for:

- ordinary double points at points of the double locus of X, which are not the Zappatic singularities of X;
- further singular points at the Zappatic singularities of X of type T_n, for $n \geqslant 3$, and Z_n, for $n \geqslant 4$,

and there exists a birational morphism $\mathcal{X}' \to \mathcal{X}$, which is the composition of blow-ups at points of the central fibre, such that $\mathcal{X}'$ is smooth.

A Zappatic degeneration will be called *good* if the central fibre is moreover a good Zappatic surface. Similarly, an embedded degeneration will be called a *planar Zappatic degeneration* if its central fibre is a planar Zappatic surface.

Notice that we require the total space $\mathcal{X}$ of a good Zappatic degeneration to be smooth at E_3-points of X.

On the other hand, if $\pi : \mathcal{X} \to \Delta$ is an arbitrary degeneration of surfaces such that $\pi^{-1}(t) = \mathcal{X}_t$, for $t \neq 0$, is by definition a smooth, irreducible and projective surface and the central fibre $\mathcal{X}_0$ is a good Zappatic surface, then the total space $\mathcal{X}$ of π may have the following singularities:

- double curves, which are double curves also for X;
- isolated double points along the double curves of X;
- further singular points at the Zappatic singularities of X, which can be isolated or may occur on double curves of the total space.

The singularities of the total space $\mathcal{X}$ of either an arbitrary degeneration with good Zappatic central fibre or of a good Zappatic degeneration will be described in details in Sections 6 and 7.

Notation 5.5. Let $\mathcal{X} \to \Delta$ be a degeneration of surfaces and let $\mathcal{X}_t$ be the general fibre, which is a smooth, irreducible and projective surface. Then, we consider the following intrinsic invariants of $\mathcal{X}_t$:

- $\chi := \chi(\mathcal{O}_{\mathcal{X}_t})$;
- $K^2 := K^2_{\mathcal{X}_t}$.

If the degeneration is assumed to be embedded in $\mathbb{P}^r$, for some r, then we also have:

- $d := \deg(\mathcal{X}_t)$;
- $g := (K+H)H/2+1$, the sectional genus of $\mathcal{X}_t$.

We will be mainly interested in computing these invariants in terms of the central fibre X. For some of them, this is quite simple. For instance, when $\mathcal{X} \to \Delta$ is an embedded degeneration in $\mathbb{P}^r$, for some r, and if the central fibre $\mathcal{X}_0 = X = \bigcup_{i=1}^v X_i$, where the X_i's are smooth, irreducible surfaces of degree d_i, $1 \leqslant i \leqslant v$, then by the flatness of the family we have

$$d = \sum_{i=1}^{v} d_i.$$

When $X = \mathcal{X}_0$ is a good Zappatic surface (in particular a good, planar Zappatic surface), we can easily compute some of the above invariants by using our results of § 3. Indeed, by Propositions 3.14 and 3.17 by the flatness of the family, we get:

Proposition 5.6. *Let $\mathcal{X} \to \Delta$ be a degeneration of surfaces and suppose that the central fibre $\mathcal{X}_0 = X = \bigcup_{i=1}^v X_i$ is a good Zappatic surface. Let $G = G_X$ be its associated graph (cf. Notation 3.12). Let C be the double locus of X, i.e. the union of the double curves of X, $C_{ij} = C_{ji} = X_i \cap X_j$ and let $c_{ij} = \deg(C_{ij})$.*
(i) If f denotes the number of (closed) faces of G, then

$$\chi = \sum_{i=1}^{v} \chi(\mathcal{O}_{X_i}) - \sum_{1 \leqslant i<j \leqslant v} \chi(\mathcal{O}_{C_{ij}}) + f. \tag{5.7}$$

Moreover, if $X = \mathcal{X}_0$ is a good, planar Zappatic surface, then

$$\chi = \chi(G) = v - e + f, \tag{5.8}$$

where e denotes the number of edges of G.
(ii) Assume further that $\mathcal{X} \to \Delta$ is embedded in $\mathbb{P}^r$. Let D be a general hyperplane section of X; let D_i be the i^{th}-smooth, irreducible component of D, which is a general hyperplane section of X_i, and let g_i be its genus. Then

$$g = \sum_{i=1}^{v} g_i + \sum_{1 \leqslant i<j \leqslant v} c_{ij} - v + 1. \tag{5.9}$$

When X is a good, planar Zappatic surface, if $G^{(1)}$ denotes the 1-skeleton of G, then:

$$g = 1 - \chi(G^{(1)}) = e - v + 1. \tag{5.10}$$

In the particular case that $\mathcal{X} \to \Delta$ is a semistable degeneration, i.e. if X has only E_3-points as Zappatic singularities and the total space $\mathcal{X}$ is smooth, then χ can be computed also in a different way by topological methods via the Clemens-Schmid's exact sequence (cf. e.g. [43]).

Proposition 5.6 is indeed more general: X is allowed to have any good Zappatic singularity, namely R_n-, S_n- and E_n-points, for any $n \geqslant 3$, the total space $\mathcal{X}$ is possibly singular, even

in dimension one, and, moreover, our computations do not depend on the fact that X is smoothable, i.e. that X is the central fibre of a degeneration.

6. Minimal and quasi-minimal singularities

In this section we shall describe the singularities that the total space of a degeneration of surfaces has at the Zappatic singularities of its central fibre. We need to recall a few general facts about reduced Cohen-Macaulay singularities and two fundamental concepts introduced and studied by Kollár in [34] and [35].

Recall that $V = V_1 \cup \cdots \cup V_r \subseteq \mathbb{P}^n$, a reduced, equidimensional and non-degenerate scheme is said to be *connected in codimension one* if it is possible to arrange its irreducible components $V_1, \ldots, V_r$ in such a way that

$$\operatorname{codim}_{V_j} V_j \cap (V_1 \cup \cdots \cup V_{j-1}) = 1, \text{ for } 2 \leqslant j \leqslant r.$$

Remark 6.1. Let X be a surface in $\mathbb{P}^r$ and C be a hyperplane section of X. If C is a projectively Cohen-Macaulay curve, then X is connected in codimension one. This immediately follows from the fact that X is projectively Cohen-Macaulay (cf. Appendix A).

Given Y an arbitrary algebraic variety, if $y \in Y$ is a reduced, Cohen-Macaulay singularity then

$$\operatorname{emdim}_y(Y) \leqslant \operatorname{mult}_y(Y) + \dim_y(Y) - 1, \tag{6.2}$$

where $\operatorname{emdim}_y(Y) = \dim(\mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2)$ is the *embedding dimension* of Y at the point y, where $\mathfrak{m}_{Y,y} \subset \mathcal{O}_{Y,y}$ denotes the maximal ideal of y in Y (see, e.g., [34]).

For any singularity $y \in Y$ of an algebraic variety Y, let us set

$$\delta_y(Y) = \operatorname{mult}_y(Y) + \dim_y(Y) - \operatorname{emdim}_y(Y) - 1. \tag{6.3}$$

If $y \in Y$ is reduced and Cohen-Macaulay, then Formula (6.2) states that $\delta_y(Y) \geqslant 0$.

Let H be any effective Cartier divisor of Y containing y. Of course one has

$$\operatorname{mult}_y(H) \geqslant \operatorname{mult}_y(Y).$$

Lemma 6.4. *(cf.* [9, Lemma 5.4]*) In the above setting, if* $\operatorname{emdim}_y(Y) = \operatorname{emdim}_y(H)$*, then* $\operatorname{mult}_y(H) > \operatorname{mult}_y(Y)$.

Proof. Let $f \in \mathcal{O}_{Y,y}$ be a local equation defining H around y. If $f \in \mathfrak{m}_{Y,y} \setminus \mathfrak{m}_{Y,y}^2$ (non-zero), then f determines a non-trivial linear functional on the Zariski tangent space $T_y(Y) \cong (\mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2)^\vee$. By the definition of $\operatorname{emdim}_y(H)$ and the fact that $f \in \mathfrak{m}_{Y,y} \setminus \mathfrak{m}_{Y,y}^2$, it follows that $\operatorname{emdim}_y(H) = \operatorname{emdim}_y(Y) - 1$. Thus, if $\operatorname{emdim}_y(Y) = \operatorname{emdim}_y(H)$, then $f \in \mathfrak{m}_{Y,y}^h$, for some $h \geqslant 2$. Therefore, $\operatorname{mult}_y(H) \geqslant h \operatorname{mult}_y(Y) > \operatorname{mult}_y(Y)$. □

We let

$$\nu := \nu_y(H) = \min\{n \in \mathbb{N} \mid f \in m_{Y,y}^n\}. \tag{6.5}$$

Notice that:

$$\operatorname{mult}_y(H) \geqslant \nu \operatorname{mult}_y(Y), \qquad \operatorname{emdim}_y(H) = \begin{cases} \operatorname{emdim}_y(Y) & \text{if } \nu > 1, \\ \operatorname{emdim}_y(Y) - 1 & \text{if } \nu = 1. \end{cases} \tag{6.6}$$

Lemma 6.7. *(cf.* [9, Lemma 5.7]*) One has*

$$\delta_y(H) \geqslant \delta_y(Y).$$

Furthermore:

(i) *if the equality holds, then either*
  (1) $\mathrm{mult}_y(H) = \mathrm{mult}_y(Y)$, $\mathrm{emdim}_y(H) = \mathrm{emdim}_y(Y) - 1$ *and* $\nu_y(H) = 1$, *or*
  (2) $\mathrm{mult}_y(H) = \mathrm{mult}_y(Y) + 1$, $\mathrm{emdim}_y(H) = \mathrm{emdim}_y(Y)$, *in which case* $\nu_y(H) = 2$ *and* $\mathrm{mult}_y(Y) = 1$;

(ii) *if* $\delta_y(H) = \delta_y(Y) + 1$, *then either*
  (1) $\mathrm{mult}_y(H) = \mathrm{mult}_y(Y) + 1$, $\mathrm{emdim}_y(H) = \mathrm{emdim}_y(Y) - 1$, *in which case* $\nu_y(H) = 1$, *or*
  (2) $\mathrm{mult}_y(H) = \mathrm{mult}_y(Y) + 2$ *and* $\mathrm{emdim}_y(H) = \mathrm{emdim}_y(Y)$, *in which case either*
   (a) $2 \leqslant \nu_y(H) \leqslant 3$ *and* $\mathrm{mult}_y(Y) = 1$, *or*
   (b) $\nu_y(H) = \mathrm{mult}_y(Y) = 2$.

Proof. It is a straightforward consequence of (6.3), of Lemma 6.4 and of (6.6). □

We will say that H has *general behaviour* at y if

$$\mathrm{mult}_y(H) = \mathrm{mult}_y(Y). \tag{6.8}$$

We will say that H has *good behaviour* at y if

$$\delta_y(H) = \delta_y(Y). \tag{6.9}$$

Notice that if H is a general hyperplane section through y, than H has both general and good behaviour.

We want to discuss in more details the relations between the two notions. We note the following facts:

Lemma 6.10. *(cf.* [9, Lemma 5.10]*) In the above setting:*

(i) *if* H *has general behaviour at* y, *then it has also good behaviour at* y;

(ii) *if* H *has good behaviour at* y, *then either*
  (1) H *has also general behaviour and* $\mathrm{emdim}_y(Y) = \mathrm{emdim}_y(H) + 1$, *or*
  (2) $\mathrm{emdim}_y(Y) = \mathrm{emdim}_y(H)$, *in which case* $\mathrm{mult}_y(Y) = 1$ *and* $\nu_y(H) = \mathrm{mult}_y(H) = 2$.

Proof. The first assertion is a trivial consequence of Lemma 6.4.

If H has good behaviour and $\mathrm{mult}_y(Y) = \mathrm{mult}_y(H)$, then it is clear that $\mathrm{emdim}_y(Y) = \mathrm{emdim}_y(H) + 1$. Otherwise, if $\mathrm{mult}_y(Y) \neq \mathrm{mult}_y(H)$, then $\mathrm{mult}_y(H) = \mathrm{mult}_y(Y) + 1$ and $\mathrm{emdim}_y(Y) = \mathrm{emdim}_y(H)$. By Lemma 6.7, (i), we have the second assertion. □

As mentioned above, we can now give two fundamental definitions (cf. [34] and [35]):

Definition 6.11. (cf. [9, Definition 5.11]) Let Y be an algebraic variety. A reduced, Cohen-Macaulay singularity $y \in Y$ is called *minimal* if the tangent cone of Y at y is geometrically reduced and $\delta_y(Y) = 0$.

Remark 6.12. Notice that if y is a smooth point for Y, then $\delta_y(Y) = 0$ and we are in the minimal case.

Definition 6.13. (cf. [9, Definition 5.13]) Let Y be an algebraic variety. A reduced, Cohen-Macaulay singularity $y \in Y$ is called *quasi-minimal* if the tangent cone of Y at y is geometrically reduced and $\delta_y(Y) = 1$.

It is important to notice the following fact:

Proposition 6.14. *(cf. [9, Proposition 5.14]) Let Y be a projective threefold and $y \in Y$ be a point. Let H be an effective Cartier divisor of Y passing through y.*

(i) *If H has a minimal singularity at y, then Y has also a minimal singularity at y. Furthermore H has general behaviour at y, unless Y is smooth at y and $\nu_y(H) = \mathrm{mult}_y(H) = 2$.*
(ii) *If H has a quasi-minimal, Gorenstein singularity at y then Y has also a quasi-minimal singularity at y, unless either*
   (1) $\mathrm{mult}_y(H) = 3$ *and* $1 \leqslant \mathrm{mult}_y(Y) \leqslant 2$, *or*
   (2) $\mathrm{emdim}_y(Y) = 4$, $\mathrm{mult}_y(Y) = 2$ *and* $\mathrm{emdim}_y(H) = \mathrm{mult}_y(H) = 4$.

Proof. Since $y \in H$ is a minimal (resp. quasi-minimal) singularity, hence reduced and Cohen-Macaulay, the singularity $y \in Y$ is reduced and Cohen-Macaulay too.

Assume that $y \in H$ is a minimal singularity, i.e. $\delta_y(H) = 0$. By Lemma 6.7, (i), and by the fact that $\delta_y(Y) \geqslant 0$, one has $\delta_y(Y) = 0$. In particular, H has good behaviour at y. By Lemma 6.10, (ii), either Y is smooth at y and $\nu_y(H) = 2$, or H has general behaviour at y. In the latter case, the tangent cone of Y at y is geometrically reduced, as is the tangent cone of H at y. Therefore, in both cases Y has a minimal singularity at y, which proves (i).

Assume that $y \in H$ is a quasi-minimal singularity, namely $\delta_y(H) = 1$. By Lemma 6.7, then either $\delta_y(Y) = 1$ or $\delta_y(Y) = 0$.

If $\delta_y(Y) = 1$, then the case (i.2) in Lemma 6.7 cannot occur, otherwise we would have $\delta_y(H) = 0$, against the assumption. Thus H has general behaviour and, as above, the tangent cone of Y at y is geometrically reduced, as the tangent cone of H at y is. Therefore Y has a quasi-minimal singularity at y.

If $\delta_y(Y) = 0$, we have the possibilities listed in Lemma 6.7, (ii). If (1) holds, we have $\mathrm{mult}_y(H) = 3$, i.e. we are in case (ii.1) of the statement. Indeed, Y is Gorenstein at y as H is, and therefore $\delta_y(Y) = 0$ implies that $\mathrm{mult}_y(Y) \leqslant 2$ by Corollary 3.2 in [49], thus $\mathrm{mult}_y(H) \leqslant 3$, and in fact $\mathrm{mult}_y(H) = 3$ because $\delta_y(H) = 1$. Also the possibilities listed in Lemma 6.7, (ii.2) lead to cases listed in the statement. □

Remark 6.15. From an analytic viewpoint, case (1) in Proposition 6.14 (*ii*), when Y is smooth at y, can be thought of as $Y = \mathbb{P}^3$ and H a cubic surface with a triple point at y.

On the other hand, case (2) can be thought of as Y being a quadric cone in $\mathbb{P}^4$ with vertex at y and as H being cut out by another quadric cone with vertex at y. The resulting singularity is therefore the cone over a quartic curve Γ in $\mathbb{P}^3$ with arithmetic genus 1, which is the complete intersection of two quadrics.

Now we describe the relation between minimal and quasi-minimal singularities and Zappatic singularities. First we need the following straightforward remark:

Lemma 6.16. *Any T_n-point (resp. Z_n-point) is a minimal (resp. quasi-minimal) surface singularity.*

The following direct consequence of Proposition 6.14 will be important for us:

Proposition 6.17. *(cf.* [9, Proposition 5.17]*) Let X be a surface with a Zappatic singularity at a point $x \in X$ and let $\mathcal{X}$ be a threefold containing X as a Cartier divisor.*

- *If x is a T_n-point for X, then x is a minimal singularity for $\mathcal{X}$ and X has general behaviour at x;*
- *If x is an E_n-point for X, then $\mathcal{X}$ has a quasi-minimal singularity at x and X has general behaviour at x, unless either:*
  (i) $\mathrm{mult}_x(X) = 3$ *and* $1 \leqslant \mathrm{mult}_x(\mathcal{X}) \leqslant 2$, *or*
  (ii) $\mathrm{emdim}_x(\mathcal{X}) = 4$, $\mathrm{mult}_x(\mathcal{X}) = 2$ *and* $\mathrm{emdim}_x(X) = \mathrm{mult}_x(X) = 4$.

In the sequel, we will need a description of a surface having as a hyperplane section a stick curve of type C_{S_n}, C_{R_n}, and C_{E_n} (cf. Examples 2.7 and 2.8).

First of all, we recall well-known results about *minimal degree surfaces* (cf. [27], page 525).

Theorem 6.18 (del Pezzo)**.** *Let X be an irreducible, non-degenerate surface of minimal degree in $\mathbb{P}^r$, $r \geqslant 3$. Then X has degree $r-1$ and is one of the following:*
(i) *a rational normal scroll;*
(ii) *the Veronese surface, if $r = 5$.*

Next we recall the result of Xambó concerning reducible minimal degree surfaces (see [53]).

Theorem 6.19 (Xambó)**.** *Let X be a non-degenerate surface which is connected in codimension one and of minimal degree in $\mathbb{P}^r$, $r \geqslant 3$. Then, X has degree $r-1$, any irreducible component of X is a minimal degree surface in a suitable projective space and any two components intersect along a line.*

In what follows, we shall frequently refer to Appendix A. Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, projectively Cohen-Macaulay surface with canonical singularities, i.e. with Du Val singularities. We recall that X is called a *del Pezzo surface* if $\mathcal{O}_X(-1) \simeq \omega_X$. We note that a del Pezzo surface is projectively Gorenstein (cf. Definition A.49 in Appendix A).

Theorem 6.20. *(del Pezzo,* [17]*. Compare also with* [9, Theorem 5.20]*) Let X be an irreducible, non-degenerate, linearly normal surface of degree r in $\mathbb{P}^r$. Then one of the following occurs:*
(i) *one has $3 \leqslant r \leqslant 9$ and X is either*
  a. *the image of the blow-up of $\mathbb{P}^2$ at $9-r$ suitable points, mapped to $\mathbb{P}^r$ via the linear system of cubics through the $9-r$ points, or*
  b. *the 2-Veronese image in $\mathbb{P}^8$ of a quadric in $\mathbb{P}^3$.*
  In each case, X is a del Pezzo surface.
(ii) *X is a cone over a smooth elliptic normal curve of degree r in $\mathbb{P}^{r-1}$.*

Proof. Let $f : Y \to X$ be a minimal desingularization of X. Let H be a general hyperplane section of X and let $C := f^*(H)$. One has $0 \leqslant p_a(H) \leqslant 1$. On the other hand, C is smooth and irreducible (by Bertini's theorem) of genus $g \leqslant p_a(H)$. By the linear normality, one has $g = 1$ and, therefore, also $p_a(H) = g = 1$. So H is smooth and irreducible, which means that X has isolated singularities.

Assume that X is not a scroll. If $r \geqslant 5$, Reider's Theorem states that $K_Y + C$ is b.p.f. on Y. Thus, $(K_Y + C)^2 \geqslant 0$. On the other hand, $(K_Y + C)C = 0$. Then, the Index Theorem implies that K_Y is numerically equivalent to $-C$. Therefore, $H^1(Y, \mathcal{O}_Y(K_Y)) = (0)$ and so Y is rational and K_Y is linearly equivalent to $-C$.

By the Adjunction Formula, the above relation is trivially true also if $r = 3$ and $r = 4$.

Now, $r = C^2 = K_Y^2 \leqslant 9$. If $r = 9$, then $Y = \mathbb{P}^2$ and $C \in |\mathcal{O}_{\mathbb{P}^2}(3)|$. If Y has no (-1)-curve, the only other possibility is $r = K_Y^2 = 8$ which leads right away to case $(i) - b$.

Suppose now that E is a (-1)-curve in Y, then $CE = 1$. We claim that $|C+E|$ is b.p.f., it contracts E and maps Y to a surface of degree $r+1$ is $\mathbb{P}^{r+1}$. Then, the assertion immediately follows by the description of the cases $r = 8$ and 9. To prove the claim, consider the exact sequence

$$0 \to \mathcal{O}_Y(C) \to \mathcal{O}_Y(C+E) \to \mathcal{O}_E \to 0$$

and remark that $h^1(\mathcal{O}_Y(C)) = h^1(\mathcal{O}_Y(-2C)) = 0$.

Now suppose that X is a scroll which is not a cone. Note that Y is an elliptic ruled surface. Let R be the pull-back via f of a line L. We claim that all curves in the algebraic system $\{R\}$ are irreducible. Otherwise, we would have some (-1)-curve on Y contracted by f, against the minimality assumption. Therefore Y is a minimal, elliptic ruled surface. Moreover, $f : Y \to X$ is finite since f cannot contract any curve transversal to R, otherwise X would be a cone, and cannot contract any curve R.

Now, $Y = \mathbb{P}_E(\mathcal{E})$, where E is an elliptic curve and $\mathcal{E}$ is a rank-two vector bundle on E. If $\mathcal{E}$ is indecomposable, then the e-invariant of the scroll is either $e = 0$ or $e = -1$ (cf. Theorem 2.15, page 377 in [30]). Let C_0 be a section of the ruling with $C_0^2 = e$. Then, $C \equiv C_0 + \alpha R$, with $\alpha \geqslant 2$.

More precisely, we have $\alpha \geqslant 3$. Otherwise, we would have $r = C_0^2 + 4 \leqslant 4$, then the hyperplane section H of X would be a complete intersection so X would be a complete intersection hence a cone. Furthermore, when $\alpha = 3$ then $e = 0$. Indeed, assume $e = -1$ and $\alpha = 3$, so $C_0 C = 2$. This would imply that X is a surface of degree 5 in $\mathbb{P}^5$ with a double line (which is the image of C_0). If we project X from this line, we have a curve of degree 3 in $\mathbb{P}^3$ which contradicts that Y is an elliptic scroll.

Notice that $K_Y \equiv -2C_0 - eR$ and therefore $K_Y - C \equiv -3C_0 - (e+\alpha)R$. Since $(C - K_Y)C_0 = \alpha + 4e$, from what observed above, in any case $(C-K_Y)C_0 > 0$. Since $(C-K_Y)^2 = 6\alpha + 15e > 0$, we have that $C - K_Y$ is big and nef. Therefore, $h^1(Y, \mathcal{O}_Y(C)) = h^1(Y, \mathcal{O}_Y(K_Y - C)) = 0$. Look now at the sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(C) \to \mathcal{O}_C(C) \to 0.$$

Since $h^1(Y, \mathcal{O}_Y) = 1$, then the restriction map

$$H^0(\mathcal{O}_Y(C)) \to H^0(\mathcal{O}_C(C))$$

is not surjective, against the hypotesis that X is a surface of degree r in $\mathbb{P}^r$.

Finally, assume that $\mathcal{E}$ is decomposable. Then, $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_i$ is a line bundle of degree d_i on E, $1 \leqslant i \leqslant 2$. Observe that $d_1 + d_2 = r$; furthermore, since X is not a cone, then $h^0(E, \mathcal{L}_i) \geqslant 2$, for each $1 \leqslant i \leqslant 2$, hence $d_i \geqslant 2$, for $1 \leqslant i \leqslant 2$. Thus, $h^0(E, \mathcal{E}) = h^0(E, \mathcal{L}_1) + h^0(E, \mathcal{L}_2) = d_1 + d_2 = r$, a contradiction. □

Since cones as in (ii) above are projectively Gorenstein surfaces (see Appendix A), the surfaces listed in Theorem 6.20 will be called *minimal Gorenstein surfaces.*

We shall make use of the following easy consequence of the Riemann-Roch theorem.

Lemma 6.21. *(Compare also with* [9, Lemma 5.21]*) Let $D \subset \mathbb{P}^r$ be a reduced (possibly reducible), non-degenerate and linearly normal curve of degree $r+d$ in $\mathbb{P}^r$, with $0 \leqslant d < r$. Then $p_a(D) = d$.*

Proof. Let $\mathcal{O}_D(H)$ be the hyperplane line bundle on D. By assumption $h^0(D, \mathcal{O}_D(H)) = r+1$ and $\deg(\mathcal{O}_D(H)) = r + d$. Riemann-Roch Theorem then gives:

$$p_a(D) = h^1(D, \mathcal{O}_D(H)) + d = h^0(D, \omega_D \otimes \mathcal{O}_D(-H)) + d, \tag{6.22}$$

where ω_D denotes the dualizing sheaf. Suppose that $h^0(\omega_D \otimes \mathcal{O}_D(-H)) > 0$. Thus, the effectiveness of $\mathcal{O}_D(H)$ and $\omega_D \otimes \mathcal{O}_D(-H)$ would imply that:

$$p_a(D) = h^0(D, \omega_D) \geqslant h^0(\mathcal{O}_D(H)) + h^0(\omega_D \otimes \mathcal{O}_D(-H)) - 1 = r + h^0(\omega_D \otimes \mathcal{O}_D(-H)),$$

which contradicts (6.22), since $d < r$ by hypothesis. □

Theorem 6.23. *(cf.* [9, Theorem 5.22]*) Let X be a non-degenerate, projectively Cohen-Macaulay surface of degree r in $\mathbb{P}^r$, $r \geqslant 3$, which is connected in codimension one. Then, any irreducible component of X is either*

(i) *a minimal Gorenstein surface, and there is at most one such component, or*
(ii) *a minimal degree surface.*

If there is a component of type (i), then the intersection in codimension one of any two distinct components can be only a line.

If there is no component of type (i), then the intersection in codimension one of any two distinct components is either a line or a (possibly reducible) conic. Moreover, if two components meet along a conic, all the other intersections are lines.

Furthermore, X is projectively Gorenstein if and only if either

(a) *X is irreducible of type (i), or*
(b) *X consists of only two components of type (ii) meeting along a conic, or*
(c) *X consists of ν, $3 \leqslant \nu \leqslant r$, components of type (ii) meeting along lines and the dual graph G_D of a general hyperplane section D of X is a cycle E_ν.*

Proof. Let D be a general hyperplane section of X. Since X is projectively Cohen-Macaulay, it is arithmetically Cohen-Macaulay (cf. Proposition A.30). This implies that D is an arithmetically Cohen-Macaulay (equiv. arithmetically normal) (equiv. arithmetically normal) curve (cf. Theorem A.31 in Appendix A). By Lemma 6.21, $p_a(D) = 1$. Therefore, for each connected subcurve D' of D, one has $0 \leqslant p_a(D') \leqslant 1$ and there is at most one irreducible component D'' with $p_a(D'') = 1$. In particular two connected subcurves of D can meet at most in two points. This implies that two irreducible components of X meet either along a line or along a conic. The linear normality of X immediately implies that each irreducible component is linearly normal too. As a consequence of Theorem 6.20 and of Lemma 6.21, all this proves the statement about the components of X and their intersection in codimension one.

It remains to prove the final part of the statement.

If X is irreducible, the assertion is trivial, so assume X reducible.

Suppose that all the intersections in codimension one of the distinct components of X are lines. If either the dual graph G_D of a general hyperplane section D of X is not a cycle or there is an irreducible component of D which is not rational, then D is not Gorenstein (see the discussion at the end of Example 2.8), contradicting the assumption that X is Gorenstein.

Conversely, if G_D is a cycle E_ν and each component of D is rational, then D is projectively Gorenstein. In particular, if all the components of D are lines, then D isomorphic to C_{E_ν} (cf. again Example 2.8). Therefore X is projectively Gorenstein too (cf. Proposition A.53 in Appendix A).

Suppose that X consists of two irreducible components meeting along a conic. Then D consists of two rational normal curves meeting at two points; thus the dualizing sheaf ω_D is trivial, i.e. D is projectively Gorenstein and Gorenstein, therefore so is X (cf. Proposition A.53 in Appendix A).

Conversely, let us suppose that X is projectively Gorenstein and there are two irreducible components X_1 and X_2 meeting along a conic. If there are other components, then there is a component X' meeting all the rest along a line. Thus, the hyperplane section contains a rational curve meeting all the rest at a point. Therefore the dualizing sheaf of D is not trivial, hence D is not Gorenstein, thus X is not Gorenstein. $\square$

By using Theorems 6.18, 6.19 and 6.20, we can prove the following result:

Proposition 6.24. *(cf.* [9, Proposition 5.23]*) Let X be a non-degenerate surface in $\mathbb{P}^r$, for some r, and let $n \geqslant 3$ be an integer.*

(i) *If $r = n+1$ and if a hyperplane section of X is C_{R_n}, then either:*
   a. *X is a smooth rational cubic scroll, possible only if $n = 3$, or*
   b. *X is a Zappatic surface, with ν irreducible components of X which are either planes or smooth quadrics, meeting along lines, and the Zappatic singularities of X are $h \geqslant 1$ points of type R_{m_i}, $i = 1, \ldots, h$, such that*

$$\sum_{i=1}^{h}(m_i - 2) = \nu - 2. \tag{6.25}$$

In particular X has global normal crossings if and only if $\nu = 2$, i.e. if and only if either $n = 3$ and X consists of a plane and a quadric meeting along a line, or $n = 4$ and X consists of two quadrics meeting along a line.

(ii) *If $r = n+1$ and if a hyperplane section of X is C_{S_n}, then either:*
   a. *X is the union of a smooth rational normal scroll $X_1 = S(1, d-1)$ of degree d, $2 \leqslant d \leqslant n$, and of $n-d$ disjoint planes each meeting X_1 along different lines of the same ruling, in which case X has global normal crossings; or*
   b. *X is planar Zappatic surface with $h \geqslant 1$ points of type S_{m_i}, $i = 1, \ldots, h$, such that*

$$\sum_{i=1}^{h}\binom{m_i - 1}{2} = \binom{n-1}{2}. \tag{6.26}$$

(iii) *If $r = n$ and if a hyperplane section of X is C_{E_n} then either:*
   a. *X is an irreducible del Pezzo surface of degree n in $\mathbb{P}^n$, possible only if $n \leqslant 6$; in particular X is smooth if $n = 6$; or*
   b. *X has two irreducible components X_1 and X_2, meeting along a (possibly reducible) conic; X_i, $i = 1, 2$, is either a smooth rational cubic scroll, or a quadric, or a plane; in particular X has global normal crossings if $X_1 \cap X_2$ is a smooth conic and neither X_1 nor X_2 is a quadric cone;*
   c. *X is a Zappatic surface whose irreducible components $X_1, \ldots, X_\nu$ of X are either planes or smooth quadrics. Moreover X has a unique E_ν-point, and no other Zappatic singularity, the singularities in codimension one being double lines.*

Proof. (i) According to Remark 6.1 and Theorem 6.19, X is connected in codimension one and is a union of minimal degree surfaces meeting along lines. Since a hyperplane section

is a C_{R_n}, then each irreducible component Y of X has to contain some line and therefore it is a rational normal scroll, or a plane. Furthermore Y has a hyperplane section which is a connected subcurve of C_{R_n}. It is then clear that Y is either a plane, or a quadric or a smooth rational normal cubic scroll.

We claim that Y cannot be a quadric cone. In fact, in this case, the hyperplane sections of Y consisting of lines pass through the vertex $y \in Y$. Since $Y \cap \overline{(X \setminus Y)}$ also consists of lines passing through y, we see that no hyperplane section of X is a C_{R_n}.

Reasoning similarly, one sees that if a component Y of X is a smooth rational cubic scroll, then Y is the only component of X, i.e. $Y = X$, which proves statement a.

Suppose now that X is reducible, so its components are either planes or smooth quadrics. The dual graph G_D of a general hyperplane section D of X is a chain of length ν and any connecting edge corresponds to a double line of X. Let $x \in X$ be a singular point and let $Y_1, \ldots, Y_m$ be the irreducible components of X containing x. Let G' be the subgraph of G_D corresponding to $Y_1 \cup \cdots \cup Y_m$. Since X is projectively Cohen-Macaulay, then clearly G' is connected, hence it is a chain. This shows that x is a Zappatic singularity of type R_m.

Finally we prove Formula (6.25). Suppose that the Zappatic singularities of X are h points $x_1, \ldots, x_h$ of type $R_{m_1}, \ldots, R_{m_h}$, respectively. Notice that the hypothesis that a hyperplane section of X is a C_{R_n} implies that two double lines of X lying on the same irreducible component have to meet at a point, because they are either lines in a plane or fibres of different rulings on a quadric.

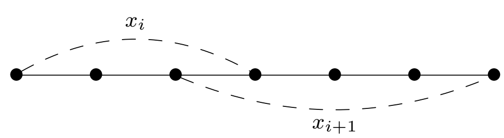


FIGURE 9. The points x_i and x_{i+1} share a common edge in the associated graph G_X.

So the graph G_X consists of h open faces corresponding to the points x_i, $1 \leqslant i \leqslant h$, and two contiguous open faces must share a common edge, as shown in Figure 9. Thus, both Formula (6.25) and the last part of statement b. immediately follow.

(ii) Arguing as in the proof of (i), one sees that any irreducible component Y of X is either a plane, or a smooth quadric or a smooth rational normal scroll with a line as a directrix.

If Y is a rational normal scroll $S(1, d-1)$ of degree $d \geqslant 2$, the subgraph of S_n corresponding to the hyperplane section of Y is S_d. Then a. follows in this case, namely all the other components of X are planes meeting Y along lines of the ruling. Note that, since X spans a $\mathbb{P}^{n+1}$, these planes are pairwise skew and therefore X has global normal crossings.

Suppose now that X is a union of planes. Then X consists of a plane Π and of $n-1$ more planes meeting Π along distinct lines. Arguing as in part (i), one sees that the planes different from Π pairwise meet only at a point in Π. Hence X is smooth off Π. On the other hand, it is clear that the singularities x_i in Π are Zappatic of type S_{m_i}, $i = 1, \ldots, h$. This corresponds to the fact that $m_i - 1$ planes different from Π pass through the same point $x_i \in \Pi$. Formula (6.26) follows by suitably counting the number of pairs of double lines in the configuration.

(iii) If X is irreducible, then a. holds by elementary properties of lines on a del Pezzo surface.

Suppose that X is reducible. Every irreducible component Y of X has a hyperplane section which is a stick curve strictly contained in C_{E_n}. By an argument we already used in part (i), then Y is either a plane, or a quadric or a smooth rational normal cubic scroll.

Suppose that an irreducible component Y meets $\overline{X \setminus Y}$ along a conic. Since C_{E_n} is projectively Gorenstein, then also X is projectively Gorenstein (cf. Proposition A.53 in Appendix A); so, by Theorem 6.23, X consists of only two irreducible components and b. follows.

Again by Theorem 6.23 and reasoning as in part (i), one sees that all the irreducible components of X are either planes or smooth quadrics and the dual graph G_D of a general hyperplane section D of X is a cycle E_ν of length ν.

As we saw in part (i), two double lines of X lying on the same irreducible component Y of X meet at a point of Y. Hence X has some singularity besides the general points on the double lines. Again, as we saw in part (i), such singularity can be either of type R_m or of type E_m, where R_m or E_n are subgraphs of the dual graph G_D of a general hyperplane section D of X. Since X is projectively Gorenstein, it has only Gorenstein singularities (cf. Remark A.52 in Appendix A), in particular R_m-points are excluded. Thus, the only singularity compatible with the above graph is an E_ν-point. $\square$

Remark 6.27. At the end of the proof of part *(iii)*, instead of using the Gorenstein property, one can prove by a direct computation that a surface X of degree n, which is a union of planes and smooth quadrics and such that the dual graph G_D of a general hyperplane section D of X is a cycle of length ν, must have an E_ν-point and no other Zappatic singularity in order to span a $\mathbb{P}^n$.

Corollary 6.28. *Let $\mathcal{X} \to \Delta$ be a degeneration of surfaces whose central fibre X is Zappatic. Let $x \in X$ be a T_n-point. Let $\mathcal{X}'$ be the blow-up of $\mathcal{X}$ at x. Let E be the exceptional divisor, let X' be the proper transform of X, $\Gamma = C_{T_n}$ be the intersection curve of E and X'. Then E is a minimal degree surface of degree n in $\mathbb{P}^{n+1} = \mathbb{P}(T_{\mathcal{X},x})$, and Γ is one of its hyperplane sections.*

In particular, if x is either a R_n- or a S_n-point, then E is as described in Proposition 6.24.

Proof. The first part of the statement directly follows from Lemma 6.16, Proposition 6.17 and Theorem 6.19. $\square$

We close this section by stating a result which will be useful in the sequel:

Corollary 6.29. *Let y be a point of a projective threefold Y. Let H be an effective Cartier divisor on Y passing through y. If H has an E_n-point at y, then Y is Gorenstein at y.*

Proof. Recall that H is Gorenstein at y (cf. Remark 3.4) and H locally behaves as a hyperplane section of Y at y (cf. the proof of Proposition 6.14), therefore Y is Gorenstein at y (cf. Theorem A.48 in Appendix A). $\square$

Remark 6.30. Let $\mathcal{X} \to \Delta$ be a degeneration of surfaces whose central fibre X is good Zappatic. From Definition 3.2 and Corollary 6.29, it follows that $\mathcal{X}$ is Gorenstein at all the points of X, except at its R_n- and S_n-points.

7. Resolutions of the total space of a degeneration of surfaces to a Zappatic one

Given $\pi : \mathcal{X} \to \Delta$ a degeneration of surfaces with good Zappatic central fibre $X = \mathcal{X}_0$, the aim of this section is to describe partial and total desingularizations of the total space $\mathcal{X}$ of

the degeneration. These will be fundamental tools in in Sections 8 and 10, where we shall combinatorially compute the K^2 of the smooth fibres of $\mathcal{X}$ (cf. Theorem 8.1) and prove the *Multiple Point Formula* (cf. Theorem 10.2), respectively, as well as in Section 9, where we shall compute the *geometric genus* of the smooth fibres of $\mathcal{X}$ (cf. Theorem 9.9).

As recalled in Remark 3.4, a good Zappatic surface X is Gorenstein only at the E_n-points, for $n \geqslant 3$. Therefore, when X is the central fibre of a degeneration $\mathcal{X} \to \Delta$, from Remark 6.30, then also $\mathcal{X}$ is Gorenstein at the E_n-points of its central fibre. Thus one can first consider a partial resolution of the total space $\mathcal{X}$ at the R_n- and S_n-points of X, for $n \geqslant 3$, in order to make both the total space and the central fibre Gorenstein. More precisely, one wants to produce a birational model of $\mathcal{X}$, denoted by

$$\mathcal{X}^G \to \Delta, \tag{7.1}$$

such that:

(i) $\mathcal{X}^G$ is isomorphic to $\mathcal{X}$ off the central fibre;
(ii) $\mathcal{X}^G$ is Gorenstein;
(iii) for each irreducible component X_i of the central fibre $\mathcal{X}_0 = X$ of $\mathcal{X}$ there is some irreducible component of the central fibre $\mathcal{X}_0^G = X^G$ of $\mathcal{X}^G$ which dominates X_i, $1 \leqslant i \leqslant v$.

$\mathcal{X}^G$ is said to be a *Gorenstein reduction* of $\mathcal{X}$. As it will be clear from the steps of Algorithm 7.2, X^G is a good Zappatic surface having only E_n-points as Zappatic singularities, so it is Gorenstein. In particular, both the dualizing sheaves $\omega_{\mathcal{X}^G}$ and ω_{X^G} will be invertible.

For our aims, we also need to completely resolve the total space of the degeneration. In this case, as it will be shown in Algorithm 7.3, one can get a *normal crossing reduction* of $\mathcal{X}$ (cf. Remark 5.2), say $\mathcal{X}^s$, such that:

(i) $\mathcal{X}^s$ is smooth and it is isomorphic to $\mathcal{X}$ off the central fibre;
(ii) its central fibre $\mathcal{X}_0^s = X^s$ is a Zappatic surface whose support is with global normal crossings, i.e. X^s_{red} is with only E_3-points as Zappatic singularities;
(iii) for each irreducible component X_i of the central fibre $\mathcal{X}_0 = X$ of $\mathcal{X}$ there is some irreducible component of the central fibre X^s which dominates X_i, $1 \leqslant i \leqslant v$.

Given a degeneration of surfaces $\pi : \mathcal{X} \to \Delta$, in order to determine both a Gorenstein and a normal crossing reduction of $\mathcal{X}$ it is necessary to carefully analyze the process, basically described in Chapter II of [33], which produces the semistable reduction.

As we said in Remark 5.2, Hironaka's result implies the existence of a normal crossing reduction of π. The birational transformation involved in resolving the singularities can be taken to be a sequence of blow-ups (which one can arrange to be at isolated points and along smooth curves) interspersed with normalization maps. For general singularities such a procedure may introduce components and double curves which can affect the computation of the invariants of the central fibre, like e.g. the ω-genus, etc.. Our next task is to show that, under the assumption that the central fibre is good Zappatic, we have very precise control over its invariants. For this we will need to more explicitly describe algorithms which produce Gorenstein and normal crossing reductions of π, respectively. In order to do this, we will use, as common in programming languages, the word "while" to indicate that the statement following it is repeated until it becomes false.

Gorenstein reduction algorithm 7.2. Let $\mathcal{X} \to \Delta$ be a degeneration of surfaces with good Zappatic central fibre. While $\mathcal{X}_0$ has a point p of type either R_n or S_n, $n \geqslant 3$, replace $\mathcal{X}$ by its blow-up at p.

Normal crossing reduction algorithm 7.3. (cf. [10, Algorithm 4.8]) Let $\mathcal{X} \to \Delta$ be a degeneration of surfaces with good Zappatic central fibre having only E_n-points, $n \geqslant 3$, as Zappatic singularities.

Step 1: while $\mathcal{X}_0$ has a point p of type E_n and $\mathcal{X}$ has multiplicity $n \geqslant 3$ at p, replace $\mathcal{X}$ by its blow-up at p;
Step 2: while $\mathcal{X}$ has a double curve γ, replace $\mathcal{X}$ by its blow-up along γ;
Step 3: if $\mathcal{X}$ has a double point p, then replace $\mathcal{X}$ by the normalization of its blow-up at p and go back to Step 2;
Step 4: while there is a component of $\mathcal{X}_0$ with a double point p, replace $\mathcal{X}$ by its blow-up at p;
Step 5: while there are two components X_1 and X_2 of $\mathcal{X}_0$ meeting along a curve with a node p, first blow-up $\mathcal{X}$ at p, then blow-up along the line which is the intersection of the exceptional divisor with the proper transform of $\mathcal{X}_0$, and finally replace $\mathcal{X}$ with the resulting threefold.

The following proposition is devoted to prove that the above algorithms work (cf. the proof of [9, Theorem 6.1] and [10, Proposition 4.9]).

Proposition 7.4. *Let $\pi : \mathcal{X} \to \Delta$ be a degeneration of surfaces with good Zappatic central fibre $X = \mathcal{X}_0 = \bigcup_{i=1}^{v} X_i$.*

(1) *Run the Gorenstein reduction algorithm 7.2; the algorithm stops after finitely many steps and its output gives a Gorenstein reduction $\pi^G : \mathcal{X}^G \to \Delta$ of π.*
(2) *Consider $\pi : \mathcal{X} \to \Delta$ such that X has only E_n-points, $n \geqslant 3$, as Zappatic singularities and run the normal crossing reduction algorithm 7.3. The algorithm stops after finitely many steps and its output gives a normal crossing reduction $\bar{\pi} : \bar{\mathcal{X}} \to \Delta$ of π.*

Proof. (1) As observed in § 5, since $\pi : \mathcal{X} \to \Delta$ is an arbitrary degeneration of smooth surfaces to a good Zappatic one, the total space $\mathcal{X}$ of π may have the following singularities:

- double curves, which are double curves also for X;
- isolated double points along the double curves of X;
- further singular points at the Zappatic singularities of X, which can be isolated or may occur on double curves of the total space.

Our aim is to prove that the Gorenstein reduction algorithm 7.2 produces a total space which is Gorenstein and a central fibre which has only E_n-points as Zappatic singularities.

By Proposition 6.17, if X has either a R_n-point or a S_n-point, $n \geqslant 3$, then the total space $\mathcal{X}$ has multiplicity n at p. Let $\mathcal{X}' \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ at a R_n-point [resp. S_n-point] p. By Proposition 6.24, the exceptional divisor E is a Zappatic surface of degree n in $\mathbb{P}^{n+1}$ such that all of its irreducible components are rational normal surfaces meeting along lines and E has at most R_m-points, $m \leqslant n$ [resp. S_m-points, $m \leqslant n$] as Zappatic singularities. Let X' be the proper transform of X. The curve $\Gamma = E \cap X'$ is a stick curve C_{R_n} [resp. C_{S_n}] which, being nodal, does not contain any Zappatic singularity of E. The new central fibre $E \cup X'$ has either E_3- or E_4-points at the double points of Γ, depending on whether E is smooth or has a double point there. These points are accordingly either smooth or double points for $\mathcal{X}'$.

The fact that the process in (1) is repeated finitely many times follows e.g. from Proposition 3.4.13 in [34]. If $\mathcal{X}^G \to \mathcal{X}$ is the composition of all the blow-ups done in (1), then $\pi^G : \mathcal{X}^G \to \Delta$ is a degeneration whose central fibre is a Zappatic surface with only E_n-points, $n \geqslant 3$, as Zappatic singularities. Thus $\pi^G : \mathcal{X}^G \to \Delta$ is a Gorenstein reduction of π (this is clear for the double points, for the E_n-points of the central fibre, see Corollary 6.29 and Remark 6.30).

(2) Our aim is now to prove that, given $\pi : \mathcal{X} \to \Delta$ as in (2), the normal crossing reduction algorithm 7.3 resolves the singularities of the total space $\mathcal{X}$ and produces a central fibre whose support has global normal crossings.

In general, the degeneration $\pi : \mathcal{X} \to \Delta$ we start with here in (2) will be the output of the Gorenstein reduction algorithm 7.2 applied to an arbitrary degeneration of smooth surfaces to a good Zappatic one.

(Step 1) By Proposition 6.17, if X has an E_n-point p, $n \geqslant 3$, then either $\mathcal{X}$ has multiplicity n at p, or $n \leqslant 4$ and $\mathcal{X}$ has at most a double point at p. In this step we consider only the former possibility, since the other cases are considered in the next steps. Let $\mathcal{X}' \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ at p. By Proposition 6.24, the exceptional divisor E is a Gorenstein surface of degree n in $\mathbb{P}^n$ which is one of the following:

(I) an irreducible del Pezzo surface, possible only if $n \leqslant 6$;

(II) a union $F = F_1 \cup F_2$ of two irreducible components F_1 and F_2 such that $F_1 \cap F_2$ is a (possibly reducible) conic; the surface F_i, $i = 1, 2$, is either a smooth rational normal cubic scroll, or a quadric, or a plane;

(III) a Zappatic surface, whose $m \leqslant n$ irreducible components meet along lines and are either planes or smooth quadrics; moreover E has a unique Zappatic singularity, which is an E_m-point.

In case (I), the del Pezzo surface E has at most isolated rational double points.

In case (II), the surface E is Zappatic unless either the conic is reducible or one of the two components is a quadric cone. Note that, if $F_1 \cap F_2$ is a conic with a double point p', then F_1 and F_2 are tangent at p' and E has not normal crossings.

Let X' be the proper transform of X. The curve $\Gamma = E \cap X'$ is a stick curve C_{E_n}. In case (II), if an irreducible component of E is a quadric cone, the vertex of the cone is a double point of Γ and $\mathcal{X}'$ also has a double point there. In case (III), the curve Γ, being nodal, does not contain the E_m-point of E. As in (1), one sees that the singular points of Γ are either smooth or double points for $\mathcal{X}'$.

In cases (I) and (II), we have eliminated the original Zappatic E_n singularity; in case (III), we have a single E_m ($m \leqslant n$) point to still consider. Whatever extra double points have been introduced, will be handled in later steps.

As in (1), also Step 1 is repeated finitely many times e.g. by Proposition 3.4.13 in [34].

(Step 2) Now the total space $\mathcal{X}$ of the degeneration has at most double points. Suppose that $\mathcal{X}$ is singular in dimension one and let γ be an irreducible curve which is double for $\mathcal{X}$. Then γ lies in the intersection of two irreducible components X_1 and X_2 of X. By Definition 3.2 of Zappatic surface and the previous steps, one has that γ is smooth and the intersection of X_1 and X_2 is transversal at the general point of γ.

Now let $\mathcal{X}' \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along γ. Let E be the exceptional divisor and X_i', $i = 1, 2$, be the proper transform of X_i in $\mathcal{X}'$. Let p be the general point of γ. Note that there

are effective Cartier divisors of X through p having a node at p. Therefore there are effective Cartier divisors of $\mathcal{X}$ through p having at p a double point of type A_k, for some $k \geqslant 1$. Since the exceptional divisor of a minimal resolution of such a point does not contain multiple components, we see that E must be reduced. Then E is a conic bundle and $\gamma_i = E \cap X'_i$, $i = 1, 2$, is a section of E isomorphic to γ.

Let C be the general ruling of E. If C is irreducible, then E is irreducible and has at most isolated double points. We remark moreover that γ_1 and γ_2 are generically smooth for the total space $\mathcal{X}'$, since they are generically smooth for E, which is a Cartier divisor of $\mathcal{X}'$. In this case, we got rid of the double curve.

Let $C = r_1 \cup r_2$ be reducible into two distinct lines. We may assume that $r_i \cap \gamma_i$, $i = 1, 2$, is a point whereas $r_i \cap \gamma_{3-i} = \emptyset$. This implies that E is reducible; one component meets X'_1 and the other meets X'_2. Hence we may write $E = F_1 \cup F_2$, where F_i meets generically transversally X'_i along γ_i, $i = 1, 2$. It may happen that F_1 and F_2 meet, generically transversally, along finitely many fibres of their rulings; away from these, they meet along the curve γ', whose general point is $r_1 \cap r_2$.

We note that γ', being isomorphic to γ, is smooth. Moreover, a local computation shows that F_1 and F_2 meet transversally at a general point of γ'. If the general point of γ' is smooth for $\mathcal{X}'$, we have nothing to do with γ', otherwise we go on blowing-up $\mathcal{X}'$ along γ'. As usual, after finitely many blow-ups we get rid of all the curves which are double for the total space.

(Step 3) Now the total space $\mathcal{X}$ of the degeneration has at most isolated double points. Let $X_{\rm red}$ be the support of the central fibre X. Note that, the first time one reaches this step, one has that $X_{\rm red} = X$, which implies that $X_{\rm red}$ is Cartier. In what follows, we only require that in a neighborhood of the singular points where we apply this step, the reduced set of components is Cartier.

By the discussion of the previous steps, one sees that a double point p of $\mathcal{X}$ can be of the following types (cf. Figure 10):

(a) an isolated double point of $X_{\rm red}$;
(b) a point of a double curve of $X_{\rm red}$;
(c) an E_3-point of $X_{\rm red}$;
(d) an E_4-point of $X_{\rm red}$;
(e) a quadruple point of $X_{\rm red}$ which lies in the intersection of three irreducible components X_1, X_2 and X_3 of $X_{\rm red}$; two of them, say X_2 and X_3, are smooth at p, whereas X_1 has a rational double point of type A_k, $k \geqslant 1$, at p. In this case, $X_2 \cup X_3$ and X_1 are both complete intersection of $\mathcal{X}$ locally at p.

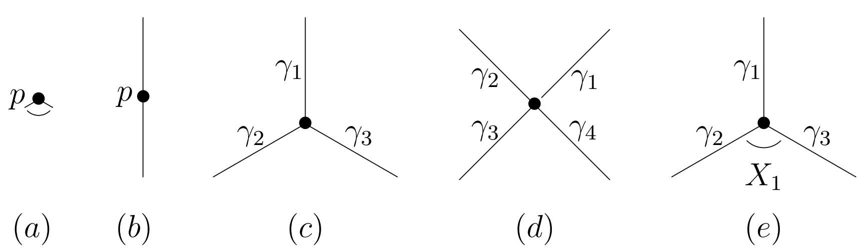


FIGURE 10. Types of double points of the total space $\mathcal{X}$.

Double points of type (a) may appear either in Step 1, if the exceptional divisor is a singular del Pezzo surface, or in Step 2, if the exceptional divisor is a singular conic bundle. In both cases, they are rational double points for X_{red}. By resolving them, one clearly gets as exceptional divisors only rational surfaces meeting each other (and the proper transform of the central fibre) along rational curves.

Consider a double point p of type (b), so p lies on a double curve which is in the intersection of two irreducible components X_1 and X_2 of X_{red}. Let $\mathcal{X}' \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ at p and let E be the exceptional divisor, which is a quadric surface in $\mathbb{P}^3$. Denote by X_i' the proper transform of X_i, $i = 1, 2$, and by p' the point $p' = E \cap X_1' \cap X_2'$. Since a general hyperplane section of $X_1 \cup X_2$ at p is a curve with a node at p, the quadric E is either:

- (i) a smooth quadric meeting X_i', $i = 1, 2$, along a line; or
- (ii) an irreducible quadric cone with vertex p'; or
- (iii) the union of two distinct planes meeting along a line γ passing through p'.

In case (i), we resolved the singularity of the total space at p. In case (ii), the new total space $\mathcal{X}'$ has an isolated double point of type (e) at p'. In case (iii), there are two possibilities: if the line γ is a double curve of $\mathcal{X}'$, then we go back to Step 2, otherwise $\mathcal{X}'$ has an isolated double point of type (d).

Let p be a double point of type (c). According to Proposition 6.17, the embedding dimension of $\mathcal{X}$ at p is 4 and the central fibre is locally analytically near p a hyperplane section of $\mathcal{X}$. Since the multiplicity of the singularity of the threefold is two, and the multiplicity of the central fibre at this point is three, the locally analytic hyperplane section must contain a component of the tangent cone of the threefold singularity. This tangent cone is therefore a quadric which has rank at most two: it is either two distinct hyperplanes or a double hyperplane (i.e. a hyperplane counted twice). In fact a local computation shows that the latter cannot happen. In the former case, when one blows up $\mathcal{X}$ at p, one introduces two planes in the new central fibre. One of these planes meets the proper transforms of the three components each in a line, forming a triangle in that plane; this plane is double in the new central fibre. (Note that at this point we introduce a non-reduced component of the central fibre; but the rest of the algorithm does not involve this multiple component.) The other of the planes, which is simple in the new central fibre, meets each of the proper transforms at a single distinct point, which is still an ordinary double point of the total space. Three more blow-ups, one each at these double points, locally resolve the total space. (This analysis follows from a local computation.)

Consider now a double point p of type (d). By Proposition 6.17, locally the tangent cone of $\mathcal{X}$ at p is a quadric cone in $\mathbb{P}^3$ and the tangent cone T of X_{red} at p is obtained by cutting it with another quadric cone in $\mathbb{P}^3$, hence T is a cone in $\mathbb{P}^4$ over a reduced, projectively normal curve of degree 4 and arithmetic genus 1 which spans a $\mathbb{P}^3$. Let $\mathcal{X}' \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ at p and let E be the exceptional divisor. Then E is a quadric meeting the proper transform of X along a stick curve C_{E_4}, therefore E is either

- (i) a smooth quadric; or
- (ii) the union of two distinct planes meeting along a line γ.

In case (i), we resolved the singularity of $\mathcal{X}$ at p. In case (ii), there are two possibilities: if the line γ is a double curve of $\mathcal{X}'$, then we go back to Step 2, otherwise $\mathcal{X}'$ has again two isolated double points of type (d) at the intersection of γ with the proper transform of X_{red}.

Here we have created double components of the central fibre, namely the exceptional divisor is counted twice. However this exceptional divisor is a Cartier divisor, and therefore X_{red} is also a Cartier divisor locally near this exceptional divisor.

Finally let $p = X_1 \cap X_2 \cap X_3$ be a double point of type (e). As in the case of type (d), locally the tangent cone of $\mathcal{X}$ at p is a quadric cone in $\mathbb{P}^3$, whereas the tangent cone T of X_{red} at p is a cone in $\mathbb{P}^4$ over a reduced, projectively normal curve of degree 4 and arithmetic genus 1 which spans a $\mathbb{P}^3$. Let $\mathcal{X}' \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ at p and let E the exceptional divisor. Denote by p' the intersection of E with the proper transform of X_2 and X_3. Then E is a quadric meeting the proper transform of X_{red} along the union of two lines and a conic spanning a $\mathbb{P}^3$, therefore E is either

(i) a smooth quadric; or
(ii) a quadric cone with vertex at p'; or
(iii) a pair of planes.

In case (i), we resolved the singularity of the total space at p. In case (ii), the total space $\mathcal{X}'$ has at p' again a point of type (e). More precisely, if p is a rational double point of type A_k, then p' is a rational double point of type A_{k-1} for E. In case (iii), the line of intersection of the two planes may be singular for the new total space; if so, we return to Step 2. If not, there are again isolated double points of type (d) and we iterate this step again.

As in the case of type (d), the reduced central fibre remains Cartier in a neighborhood of the new exceptional locus.

It is clear that, after having repeated finitely many times Steps 2 and 3, one resolves the singularities of the total space at the double points of these five types (a)-(e).

We remark that we can proceed, in Step 3, by first resolving all of the points of type (c), and that such points are not created in the resolutions of points of type (d) and (e). In fact they are not created in any later step of the algorithm. Indeed, anytime three components X_1, X_2, and X_3 concur at a point as in type (c) where at least one of the three surfaces has been created by blowing-up, we claim that exactly one of the three surfaces has been created by blowing-up (i.e., is an exceptional divisor). Since such an exceptional divisor is locally Cartier and smooth at the point, then the total space is smooth at the point and therefore the point cannot be of type (c). To prove the claim, note that the only other possibility is that two of the three components, say X_2 and X_3 belong to an exceptional divisor. By blowing them down, then X_1 acquires a singular point which is worse than an ordinary double point, which is impossible.

(Step 4) Let p be an isolated double point of the central fibre X which is a smooth point of $\mathcal{X}$. According to the previous steps, p is either a rational double point of a del Pezzo surface or the singular point of a reduced fibre of a conic bundle. In both cases, the singularity of X at p is resolved by finitely many blow-ups. Since p is a smooth point of $\mathcal{X}$, the exceptional divisor of each blow-up is a plane.

(Step 5) Following the previous steps, one sees that the support of the central fibre X has global normal crossings, except at the points p, where two components X_1 and X_2 of X meet along a curve with a node at p. Note that X_1 and X_2 are indeed tangent at p.

If one blows-up $\mathcal{X}$ at p, the exceptional divisor E is a plane meeting the proper transform X_i' of X_i, $i = 1, 2$, along a line γ, which is a (-1)-curve both on X_1' and X_2'. The support

of the new central fibre has not yet normal crossings. However a further blow-up along γ produces the normal crossing reduction. $\square$

8. Combinatorial computation of K^2

The results contained in Sections 6 and 7 will be used in this section to prove combinatorial formulas for $K^2 = K^2_{\mathcal{X}_t}$, where $\mathcal{X}_t$ is a smooth surface which degenerates to a good Zappatic surface $\mathcal{X}_0 = X = \bigcup_{i=1}^v X_i$, i.e. $\mathcal{X}_t$ is the general fibre of a degeneration of surfaces whose central fibre is good Zappatic (cf. Notation 5.5).

Indeed, by using the combinatorial data associated to X and G_X (cf. Definition 3.6 and Notation 3.12), we shall prove the following main result:

Theorem 8.1. *(cf.* [9, Theorem 6.1]*) Let $\mathcal{X} \to \Delta$ be a degeneration of surfaces whose central fibre is a good Zappatic surface $X = \mathcal{X}_0 = \bigcup_{i=1}^v X_i$. Let $C_{ij} = X_i \cap X_j$ be a double curve of X, which is considered as a curve on X_i, for $1 \leqslant i \neq j \leqslant v$.*

If $K^2 := K^2_{\mathcal{X}_t}$, for $t \neq 0$, then (cf. Notation 3.12):

$$K^2 = \sum_{i=1}^{v} \left(K^2_{X_i} + \sum_{j \neq i} (4g_{ij} - C^2_{ij}) \right) - 8e + \sum_{n \geqslant 3} 2n f_n + r_3 + k, \tag{8.2}$$

where k depends only on the presence of R_n- and S_n-points, for $n \geqslant 4$, and precisely:

$$\sum_{n \geqslant 4} (n-2)(r_n + s_n) \leqslant k \leqslant \sum_{n \geqslant 4} \left((2n-5) r_n + \binom{n-1}{2} s_n \right). \tag{8.3}$$

In case $\mathcal{X}$ is an embedded degeneration and X is also planar, we have the following:

Corollary 8.4. *Let $\mathcal{X} \to \Delta$ be an embedded degeneration of surfaces whose central fibre is a good planar Zappatic surface $X = \mathcal{X}_0 = \bigcup_{i=1}^v \Pi_i$. Then:*

$$K^2 = 9v - 10e + \sum_{n \geqslant 3} 2n f_n + r_3 + k \tag{8.5}$$

where k is as in (8.3) *and depends only on the presence of R_n- and S_n-points, for $n \geqslant 4$.*

Proof. Clearly $g_{ij} = 0$, for each $1 \leqslant i \neq j \leqslant v$, whereas $C^2_{ij} = 1$, for each pair (i,j) s.t. $e_{ij} \in E$, otherwise $C^2_{ij} = 0$. $\square$

The proof of Theorem 8.1 will be done in several steps. The first one is to compute K^2 when X has only E_n-points. In this case, and only in this case, K_X is a Cartier divisor (cf. Remark 3.4).

Theorem 8.6. *(cf.* [9, Theorem 6.6]*) Under the assumptions of Theorem 8.1, if $X = \bigcup_{i=1}^v X_i$ has only E_n-points, for $n \geqslant 3$, then:*

$$K^2 = \sum_{i=1}^{v} \left(K^2_{X_i} + \sum_{j \neq i} (4g_{ij} - C^2_{ij}) \right) - 8e + \sum_{n \geqslant 3} 2n f_n. \tag{8.7}$$

Proof. Recall that, in this case, the total space $\mathcal{X}$ is Gorenstein (cf. Remark 6.30). Thus, $K_{\mathcal{X}}$ is a Cartier divisor on $\mathcal{X}$. Therefore K_X is also Cartier and it makes sense to consider K^2_X and the adjunction formula states $K_X = (K_{\mathcal{X}} + X)_{|X}$.

We claim that

$$K_{X|X_i} = (K_{\mathcal{X}} + X)_{|X_i} = K_{X_i} + C_i, \tag{8.8}$$

where $C_i = \sum_{j \neq i} C_{ij}$ is the union of the double curves of X lying on the irreducible component X_i, for each $1 \leqslant i \leqslant v$. Since $\mathcal{O}_X(K_X)$ is invertible, it suffices to prove (8.8) off the E_n-points. In other words, we can consider the surfaces X_i as if they were Cartier divisors on $\mathcal{X}$. Then, we have:

$$K_{X|X_i} = (K_{\mathcal{X}} + X)_{|X_i} = \big(K_{\mathcal{X}} + X_i + \sum_{j \neq i} X_j\big)_{|X_i} = K_{X_i} + C_i, \tag{8.9}$$

as we had to show. Furthermore:

$$\begin{aligned} K^2 &= (K_{\mathcal{X}} + \mathcal{X}_t)^2 \cdot \mathcal{X}_t = (K_{\mathcal{X}} + X)^2 \cdot X = (K_{\mathcal{X}} + X)^2 \cdot \sum_{i=1}^{v} X_i = \sum_{i=1}^{v} \big((K_{\mathcal{X}} + X)_{|X_i}\big)^2 = \\ &= \sum_{i=1}^{v} (K_{X_i}^2 + 2C_i K_{X_i} + C_i^2) = \sum_{i=1}^{v} K_{X_i}^2 + \sum_{i=1}^{v} C_i K_{X_i} + \sum_{i=1}^{v} C_i(C_i + K_{X_i}) = \\ &= \sum_{i=1}^{v} K_{X_i}^2 + \sum_{i=1}^{v} (\sum_{j \neq i} C_{ij}) K_{X_i} + \sum_{i=1}^{v} 2(p_a(C_i) - 1). \end{aligned} \tag{8.10}$$

As in Notation 3.12, $C_{ij} = \sum_{t=1}^{h_{ij}} C_{ij}^t$ is the sum of its disjoint, smooth, irreducible components, where h_{ij} is the number of these components. Thus,

$$C_{ij} K_{X_i} = \sum_{t=1}^{h_{ij}} (C_{ij}^t K_{X_i}),$$

for each $1 \leqslant i \neq j \leqslant v$. If we denote by g_{ij}^t the geometric genus of the smooth, irreducible curve C_{ij}^t, by the Adjunction Formula on each C_{ij}^t, we have the following intersection number on the surface X_i:

$$C_{ij} K_{X_i} = \sum_{t=1}^{h_{ij}} (2g_{ij}^t - 2 - (C_{ij}^t)^2) = 2g_{ij} - 2h_{ij} - C_{ij}^2,$$

where the last equality follows from the definition of geometric genus of C_{ij} and the fact that $C_{ij}^s C_{ij}^t = 0$, for any $1 \leqslant t \neq s \leqslant h_{ij}$.

Therefore, by the distributivity of the intersection form and by (8.10), we get:

$$K^2 = \sum_{i=1}^{v} K_{X_i}^2 + \sum_{i=1}^{v} (\sum_{j \neq i} (2g_{ij} - 2h_{ij}) - C_{ij}^2) + \sum_{i=1}^{v} 2(p_a(C_i) - 1). \tag{8.11}$$

For each index i, consider now the normalization $\nu_i : \tilde{C}_i \to C_i$ of the curve C_i lying on X_i; this determines the short exact sequence:

$$0 \to \mathcal{O}_{C_i} \to (\nu_i)_*(\mathcal{O}_{\tilde{C}_i}) \to \underline{t}_i \to 0, \tag{8.12}$$

where $\underline{t}_i$ is a sky-scraper sheaf supported on $\mathrm{Sing}(C_i)$, as a curve in X_i. By using Notation 3.12, the long exact sequence in cohomology induced by (8.12) gives that:

$$\chi(\mathcal{O}_{C_i}) + h^0(\underline{t}_i) = \sum_{j\neq i}\sum_{t=1}^{h_{ij}} \chi(\mathcal{O}_{C_{ij}^t}) = \sum_{j\neq i}(h_{ij} - g_{ij}).$$

Since $\chi(\mathcal{O}_{C_i}) = 1 - p_a(C_i)$, we get

$$p_a(C_i) - 1 = \sum_{j\neq i}(g_{ij} - h_{ij}) + h^0(\underline{t}_i),\ 1 \leqslant i \leqslant v. \tag{8.13}$$

By plugging Formula (8.13) in (8.11), we get:

$$K^2 = \sum_{i=1}^{v}\left(K_{X_i}^2 + \sum_{j\neq i}(4g_{ij} - C_{ij}^2)\right) - 8e + 2\sum_{i=1}^{v} h^0(\underline{t}_i). \tag{8.14}$$

To complete the proof, we need to compute $h^0(\underline{t}_i)$. By definition of $\underline{t}_i$, this computation is a local problem. Suppose that p is an E_n-point of X lying on X_i, for some i. By the very definition of E_n-point (cf. Definition 3.1 and Example 2.8), p is a node for the curve $C_i \subset X_i$; therefore $h^0(\underline{t}_{i|p}) = 1$. The same holds on each of the other $n-1$ curves $C_j \subset X_j$, $1 \leqslant j \neq i \leqslant n$, concurring at the E_n-point p. Therefore, by (8.14), we get (8.7). □

Proof of Theorem 8.1. The previous argument proves that, in this more general case, one has:

$$K^2 = \sum_{i=1}^{v}\left(K_{X_i}^2 + \sum_{j\neq i}(4g_{ij} - C_{ij}^2)\right) - 8e + 2\sum_{i=1}^{v} h^0(\underline{t}_i) - c \tag{8.15}$$

where c is a positive correction term which depends only on the points where $\mathcal{X}$ is not Gorenstein, i.e. at the R_n- and S_n-points of its central fibre X.

To prove the statement, we have to compute:

(i) the contribution of $h^0(\underline{t}_i)$ given by the R_n- and the S_n-points of X, for each $1 \leqslant i \leqslant v$;
(ii) the correction term c.

For (i), suppose first that p is a R_n-point of X and let C_i be one of the curves passing through p. By definition (cf. Example 2.7), the point p is either a smooth point or a node for $C_i \subset X_i$. In the first case we have $h^0(\underline{t}_{i|p}) = 0$ whereas, in the latter, $h^0(\underline{t}_{i|p}) = 1$. More precisely, among the n indexes involved in the R_n-point there are exactly two indexes, say i_1 and i_n, such that C_{i_j} is smooth at p, for $j = 1$ and $j = n$, and $n-2$ indexes such that C_{i_j} has a node at p, for $2 \leqslant j \leqslant n-1$. On the other hand, if we assume that p is a S_n-point, then p is an ordinary $(n-1)$-tuple point for only one of the curves concurring at p, say $C_i \subset X_i$, and a simple point for all the other curves $C_j \subset X_j$, $1 \leqslant j \neq i \leqslant n$. Recall that an ordinary $(n-1)$-tuple point contributes $\binom{n-1}{2}$ to $h^0(\underline{t}_i)$.

Therefore, from (8.15), we have:

$$K^2 = \sum_{i=1}^{v}\left(K_{X_i}^2 + \sum_{j\neq i}(4g_{ij} - C_{ij}^2)\right) - 8e + \sum_{n\geqslant 3} 2nf_n + \sum_{n\geqslant 3} 2(n-2)r_n + \sum_{n\geqslant 4}(n-1)(n-2)s_n - c.$$

In order to compute the correction term c, we have to perform a partial resolution of $\mathcal{X}$ at the R_n- and S_n-points of X, which makes the total space Gorenstein; i.e. we have to consider

a Gorenstein reduction of the degeneration $\mathcal{X} \to \Delta$.This will give us (8.2), i.e.

$$K^2 = \sum_{i=1}^{v} \left(K_{X_i}^2 + \sum_{j \neq i} (4g_{ij} - C_{ij}^2) \right) - 8e + \sum_{n \geqslant 3} 2nf_n + r_3 + k,$$

where

$$k := \sum_{n \geqslant 3} 2(n-2)r_n \ - r_3 \ + \sum_{n \geqslant 4} (n-1)(n-2)s_n \ - c.$$

From Algorithm 7.2 and Proposition 7.4 - (1), we know that $\mathcal{X} \to \Delta$ admits a Gorenstein reduction; we now consider a detailed analysis of this Gorenstein reduction in order to compute the contribution c. It is clear that the contribution to c of each such point is purely local. In other words,

$$c = \sum_x c_x$$

where x varies in the set of R_n- and S_n-points of X and where c_x is the contribution at x to the computation of K^2 as above.

In the next Proposition 8.16, we shall compute such local contributions. This result, together with Theorem 8.6, will conclude the proof. □

Proposition 8.16. *In the hypothesis of Theorem 8.1, if $x \in X$ is a R_n-point then:*

$$n - 2 \geqslant c_x \geqslant 1,$$

whereas if $x \in X$ is a S_n-point then:

$$(n-2)^2 \geqslant c_x \geqslant \binom{n-1}{2}.$$

Proof. Since the problem is local, we may (and will) assume that $\mathcal{X}$ is Gorenstein, except at a point x, and that each irreducible component X_i of X passing through x is a plane, denoted by Π_i.

First we will deal with the case $n = 3$.

Claim 8.17. *If x is a R_3-point, then*

$$c_x = 1.$$

Proof of the claim. From Proposition 7.4-(1), let us blow-up the point $x \in \mathcal{X}$ as in Corollary 6.28.

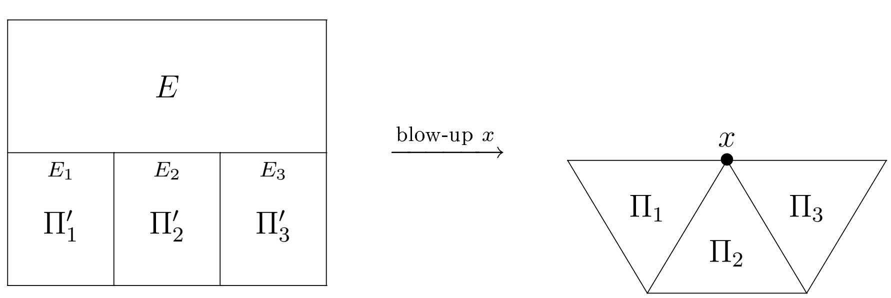


FIGURE 11. Blowing-up a R_3-point x.

We get a new total space $\mathfrak{X}'$. We denote by E the exceptional divisor, by Π'_i the proper transform of Π_i and by $X' = \cup \Pi'_i$ the proper transform of X, as in Figure 11. We remark that the three planes Π_i, $i = 1, 2, 3$, concurring at x, are blown-up in this process, whereas the remaining planes stay untouched. We call E_i the exceptional divisor on the blown-up plane Π_i. Let $\Gamma = E_1 + E_2 + E_3$ be the intersection curve of E and X'. By Corollary 6.28, E is a non-degenerate surface of degree 3 in $\mathbb{P}^4$, with Γ as a hyperplane section.

Suppose first that E is irreducible. Then $\mathfrak{X}'$ is Gorenstein and by adjunction:

$$K^2 = (K_{X'} + \Gamma)^2 + (K_E + \Gamma)^2. \tag{8.18}$$

Since E is a rational normal cubic scroll in $\mathbb{P}^4$, then:

$$(K_E + \Gamma)^2 = 1, \tag{8.19}$$

whereas the other term is:

$$(K_{X'} + \Gamma)^2 = \sum_i (K_{X'|\Pi'_i} + \Gamma_{\Pi'_i})^2 = \sum_{i=1}^{3} (K_{X'|\Pi'_i} + E_i)^2 + \sum_{j \geqslant 4} K^2_{X'|\Pi'_j}.$$

Reasoning as in the proof of Theorem 8.6, one sees that

$$\sum_{j \geqslant 4} K^2_{X'|\Pi'_j} = \sum_{j \geqslant 4} (w_j - 3)^2.$$

On the other hand, one has

$$(K_{X'|\Pi'_i} + E_i)^2 = (w_i - 3)^2 - 1, \quad i = 1, 3, \qquad (K_{X'|\Pi'_2} + E_2)^2 = (w_2 - 3)^2.$$

Putting all together, it follows that $c_x = 1$.

Suppose now that E is reducible and X' is still Gorenstein. In this case E is as described in Proposition 6.24 (ii), b, and in Corollary 6.28 and the proof proceeds as above, once one remarks that (8.19) holds. This can be left to the reader to verify (see Figure 12).

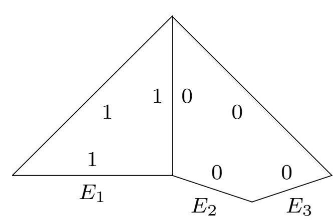


FIGURE 12. E splits in a plane and a quadric.

Suppose that E is reducible and X' is not Gorenstein. This means that E consists of a cone over a C_{R_3} with vertex x', hence x' is again a R_3-point. Therefore, as in Proposition 7.4-(1), we have to repeat the process by blowing-up x'. After finitely many steps this procedure stops (cf. e.g. Proposition 3.4.13 in [34]). In order to conclude the proof in this case, one has simply to remark that no contribution to K^2 comes from the surfaces created in the intermediate steps.

To see this, it suffices to make this computation when only two blow-ups are needed. This is the situation showed in Figure 13 where:

- $\mathfrak{X}'' \to \mathfrak{X}'$ is the blow-up at x',

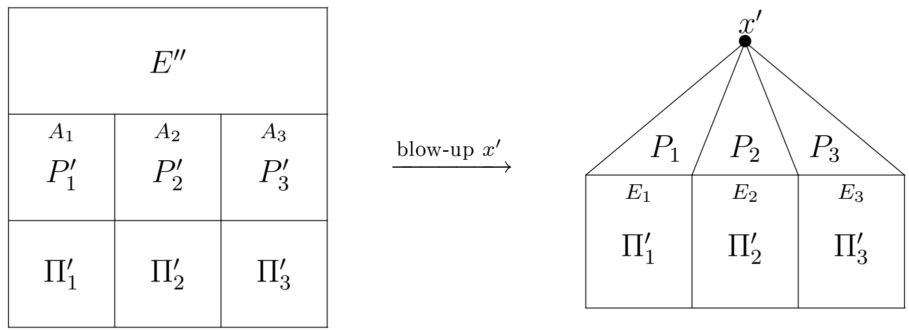


FIGURE 13. blowing-up a R_3-point x' infinitely near to the R_3-point x

- $X' = \sum \Pi'_i$ the proper transform of X' on $\mathfrak{X}''$,
- $E' = P'_1 + P'_2 + P'_3$ is the strict transform of $E = P_1 + P_2 + P_3$ on $\mathfrak{X}''$,
- E'' is the exceptional divisor of the blow-up.

We remark that P'_i, $i = 1, 2, 3$, is the blow-up of the plane P_i. We denote by Λ_i the pullback to P'_i of a line, and by A_i the exceptional divisor of P'_i. Then their contributions to the computation of K^2 are:

$$(K_{P'_i} + \Lambda_i + (\Lambda_i - A_i) + A_i)^2 = (-\Lambda_i + A_i)^2 = 0, \quad i = 1, 3,$$
$$(K_{P'_2} + \Lambda_2 + 2(\Lambda_2 - A_2) + A_2)^2 = 0.$$

This concludes the proof of Claim 8.17. □

Consider now the case that $n = 4$ and x is a R_4-point.

Claim 8.20. *If x is a R_4-point, then*

$$2 \geqslant c_x \geqslant 1.$$

Proof of the claim. As before, we blow-up the point $x \in \mathfrak{X}$; let $\mathfrak{X}'$ be the new total space and let E be the exceptional divisor. By Corollary 6.28, E is a non-degenerate surface of minimal degree in $\mathbb{P}^5$ with $\Gamma = E_1 + E_2 + E_3 + E_4$ as a hyperplane section. By Proposition 6.24, E is reducible and the following cases may occur:

(i) E has global normal crossings, in which case E consists of two quadrics Q_1, Q_2 meeting along a line (see Figure 14);

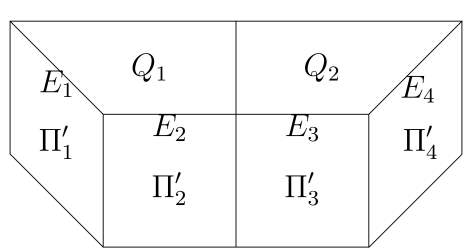
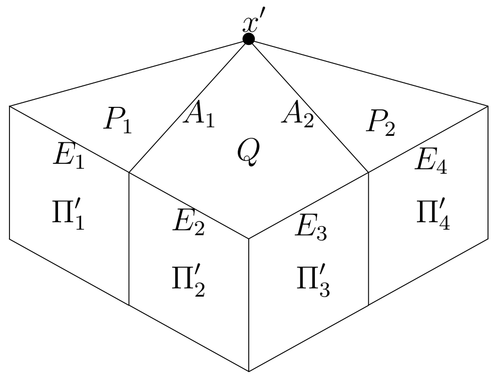
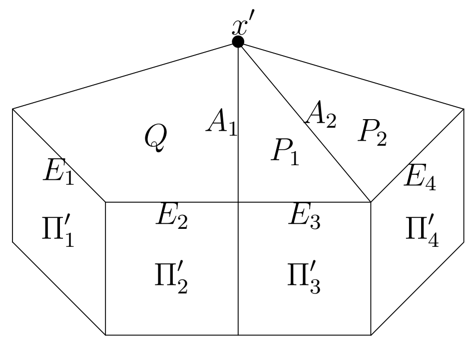


FIGURE 14. The exceptional divisor E has global normal crossings.



a) The quadric in the middle



b) The quadric on one side

FIGURE 15. E consists of a quadric and two planes and has a R_3-point x'.

(ii) E has one R_3-point x', in which case E consists of a quadric Q and two planes P_1, P_2 (see Figure 15);

(iii) E has two R_3-points x', x'', in which case E consists of four planes $P_1, \ldots, P_4$, i.e. a planar Zappatic surface whose associated graph is the tree R_4 (see Figure 16);

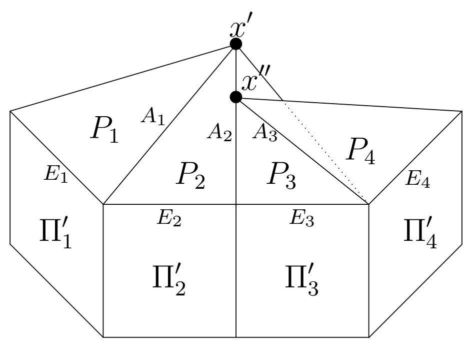


FIGURE 16. E consists of four planes and has two R_3-points x', x''.

(iv) E has one R_4-point x', in which case E consists of four planes, i.e. a planar Zappatic surface whose associated graph is an open 4-face (cf. Figures 5, 6 and 17).

In case (i), $\mathfrak{X}'$ is Gorenstein and we can compute K^2 as we did in the proof of Claim 8.17. Formula (8.18) still holds and one has $(K_E + \Gamma)^2 = 0$, whereas:

$$(8.21)\quad (K_{X'}+\Gamma)^2 = \sum_i (K_{X'|\Pi'_i}+\Gamma_{\Pi'_i})^2 = \sum_{i=1}^{4}(K_{X'|\Pi'_i}+E_i)^2 + \sum_{j\geqslant 4} K^2_{X'|\Pi'_j} = \sum_{j\geqslant 1}(w_j-3)^2-2,$$

because the computations on the blown-up planes $\Pi'_1, \ldots, \Pi'_4$ give:

$$(K_{X'|\Pi'_i} + E_i)^2 = (w_i-3)^2 - 1, \quad i = 1, 4, \qquad (K_{X'|\Pi'_i} + E_i)^2 = (w_i-3)^2, \quad i = 2, 3.$$

This proves that $c_x = 2$ in this case.

In case (ii), there are two possibilities corresponding to cases (a) and (b) of Figure 15. Let us first consider the former possibility. By Claim 8.17, in order to compute K^2 we have to add up three quantities:

- the contribution of $(K_{X'}+\Gamma)^2$, which is computed in (8.21);
- the contribution to K^2 of E, as if E had only global normal crossings, i.e.:

$$(K_{P_1}+A_1+E_1)^2+(K_{P_2}+A_2+E_4)^2+(K_Q+A_1+A_2+E_2+E_3)^2=2$$

- the contribution of the R_3-point x', which is $c_{x'}=1$ by Claim 8.17.

Putting all this together, it follows that $c_x=1$ in this case. Consider now the latter possibility, i.e. suppose that the quadric meets only one plane. We can compute the three contributions to K^2 as above: the contribution of $(K_{X'}+\Gamma)^2$ and of the R_3-point x' do not change, whereas the contribution to K^2 of E, as if E had only global normal crossings, is:

$$(K_Q+A_1+E_1+E_2)^2+(K_{P_1}+A_1+A_2+E_3)^2+(K_{P_4}+A_3+E_4)^2=1,$$

therefore we find that $c_x=2$, which concludes the proof for case (ii).

In case (iii), we use the same strategy as in case (ii), namely we add up $(K_{X'}+\Gamma)^2$, the contribution to K^2 of E, as if E had only global normal crossings, which turns out to be 2, and then subtract 2, because of the contribution of the two R_3-points x', x''. Summing up, one finds $c_x=2$ in this case.

In case (iv), we have to repeat the process by blowing-up x', see Figure 17. After finitely many steps (cf. e.g. Proposition 3.4.13 in [34]), this procedure stops in the sense that the exceptional divisor will be as in case (i), (ii) or (iii).

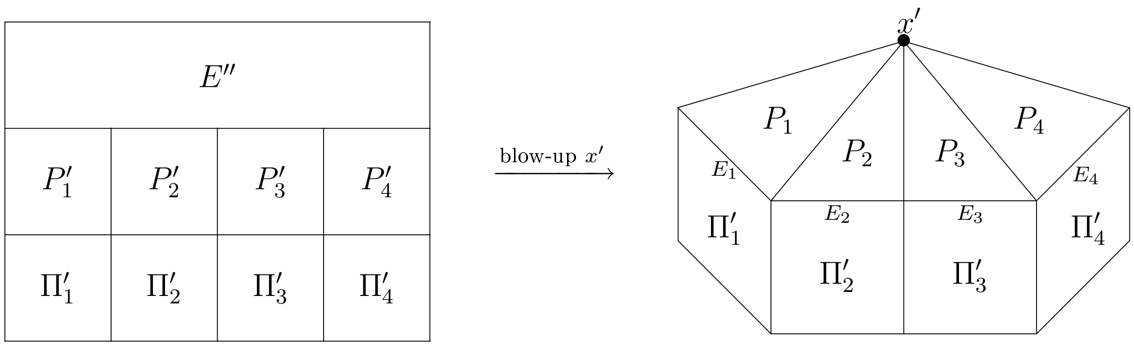


FIGURE 17. Blowing-up a R_4-point x' infinitely near to x.

In order to conclude the proof of Claim 8.20, one has to remark that no contribution to K^2 comes from the surfaces created in the intermediate steps (the blown-up planes P'_i in Figure 17). This can be done exactly in the same way as we did in the proof of Claim 8.17. □

Remark 8.22. The proof of Claim 8.20 is purely combinatorial. However there is a nice geometric motivation for the two cases $c_x=2$ and $c_x=1$, when x is a R_4-point, which resides in the fact that the local deformation space of a R_4-point is reducible. This corresponds to the fact that the cone over C_{R_4} can be smoothed in both a Veronese surface and a rational normal quartic scroll, which have $K^2=9$ and $K^2=8$, respectively.

Consider now the case that x is a R_n-point.

Claim 8.23. *If x is a R_n-point, then*

$$n-2 \geqslant c_x \geqslant 1. \tag{8.24}$$

Proof of the claim. The claim for $n = 3, 4$ has already been proved, so we assume $n \geqslant 5$ and proceed by induction on n. As usual, we blow-up the point $x \in \mathcal{X}$.

By Corollary 6.28, the exceptional divisor E is a non-degenerate surface of minimal degree in $\mathbb{P}^{n+1}$ with $\Gamma = E_1 + \ldots + E_n$ as a hyperplane section. By Proposition 6.24, E is reducible and the following cases may occur:

(i) E consists of $\nu \geqslant 3$ irreducible components $P_1, \ldots, P_\nu$, which are either planes or smooth quadrics, and E has h Zappatic singular points $x_1, \ldots, x_h$ of type $R_{m_1}, \ldots, R_{m_h}$ such that $m_i < n$, $i = 1, \ldots, h$;

(ii) E has one R_n-point x', in which case E consists of n planes, i.e. a planar Zappatic surface whose associated graph is an open n-face.

In case (ii), one has to repeat the process by blowing-up x'. After finitely many steps (cf. e.g. Proposition 3.4.13 in [34]), the exceptional divisor will necessarily be as in case (i). We remark that no contribution to K^2 comes from the surfaces created in the intermediate steps, as one can prove exactly in the same way as we did in the proof of Claim 8.17.

Thus, it suffices to prove the statement for the case (i). Notice that $\mathcal{X}'$ is not Gorenstein, nonetheless we can compute K^2 since we know (the upper and lower bounds of) the contribution of x_i by induction. We can indeed proceed as in case (ii) of the proof of Claim 8.20, namely, we have to add up three quantities:

- the contribution of $(K_{X'} + \Gamma)^2$;
- the contribution to K^2 of E, as if E had only global normal crossings;
- the contributions of the points x_i which is known by induction.

Let us compute these contributions. As for the first one, one has:

$$(K_{X'} + \Gamma)^2 = \sum_{i=1}^{n} (K_{X'|\Pi'_i} + E_i)^2 + \sum_{j \geqslant n} K^2_{X'|\Pi'_j} = \sum_{j \geqslant 1} (w_j - 3)^2 - 2,$$

since the computations on the blown-up planes $\Pi'_1, \ldots, \Pi'_n$ give:

$$(K_{X'|\Pi'_i} + E_i)^2 = (w_i - 3)^2 - 1, \quad i = 1, n,$$
$$(K_{X'|\Pi'_i} + E_i)^2 = (w_i - 3)^2, \quad 2 \leqslant i \leqslant n - 1.$$

In order to compute the second contribution, one has to introduce some notation, precisely we let:

- $P_1, \ldots, P_\nu$ be the irreducible components of E, which are either planes or smooth quadrics, ordered in such a way that the intersections in codimension one are as follows: P_i meets P_{i+1}, $i = 1, \ldots, \nu - 1$, along a line;
- A_i be the line which is the intersection of P_i and P_{i+1};
- $\varepsilon_i = \deg(P_i) - 1$, which is 0 if P_i is a plane and 1 if P_i is a quadric;
- $j(i) = i + \sum_{k=1}^{i-1} \varepsilon_j$. With this notation, if P_i is a plane, it meets the blown-up plane $\Pi'_{j(i)}$ along $E_{j(i)}$, whereas if P_i is a quadric, it meets the blown-up planes $\Pi'_{j(i)}$ and $\Pi'_{j(i)+1}$ along $E_{j(i)}$ and $E_{j(i)+1}$, respectively.

Then the contribution to K^2 of E, as if E had only global normal crossings, is:

$$(K_{P_1}+A_1+E_1+\varepsilon_1E_2)^2+(K_{P_\nu}+A_{\nu-1}+\varepsilon_\nu E_{n-1}+E_n)^2+ \\ +\sum_{i=2}^{\nu-1}(K_{P_i}+A_{i-1}+A_i+E_{j(i)}+\varepsilon_iE_{j(i)+1})^2=2-\varepsilon_1-\varepsilon_\nu.$$

Finally, by induction, the contribution $\sum_{i=1}^h c_{x_i}$ of the points x_i is such that:

$$\nu-2=\sum_{i=1}^h(m_i-2)\geqslant\sum_{i=1}^h c_{x_i}\geqslant\sum_{i=1}^h 1=h,$$

where the first equality is just (6.25).

Putting all this together, it follows that:

$$c_x=\varepsilon_1+\varepsilon_\nu+\sum_{i=1}^h c_{x_i},$$

hence an upper bound for c_x is

$$c_x\leqslant\varepsilon_1+\varepsilon_\nu+\nu-2\leqslant n-2,$$

because $n=\nu+\sum_{i=1}^\nu\varepsilon_i$, whereas a lower bound is

$$c_x\geqslant\varepsilon_1+\varepsilon_\nu+h\geqslant h\geqslant 1, \tag{8.25}$$

which concludes the proof of Claim 8.23. □

Remark 8.26. If $c_x=1$, then in (8.25) all inequalities must be equalities, thus $h=1$ and $\varepsilon_1=\varepsilon_\nu=0$. This means that there is only one point x_1 infinitely near to x, of type R_ν, and that the external irreducible components of E, i.e. P_1 and P_ν, are planes. There is no combinatorial obstruction to this situation.

For example, let x be a R_n-point such that the exceptional divisor E consists of $\nu=n-1$ irreducible components, namely $n-2$ planes and a quadric adjacent to two planes, forming a R_{n-1}-point x'. By the proof of Claim 8.20 (case (ii), former possibility), it follows that $c_x=c_{x'}$. Since, as we saw, the contribution of a R_4-point can be 1, by induction we may have that also a R_n-point contributes by 1.

From the proof of Claim 8.23, it follows that the upper bound $c_x=n-2$ is attained when for example the exceptional divisor E consists of n planes forming $n-2$ points of type R_3.

More generally, one can see that there is no combinatorial obstruction for c_x to attain any possible value between the upper and lower bounds in (8.24).

Finally, consider the case that x is of type S_n.

Claim 8.27. *If x is a S_n-point, then*

$$(n-2)^2\geqslant c_x\geqslant\binom{n-1}{2}. \tag{8.28}$$

Proof. We remark that we do not need to take care of 1-dimensional singularities of the total space of the degeneration, as we have already noted in Claim 8.23.

Notice that $S_3=R_3$ and, for $n=3$, Formula (8.28) trivially follows from Claim 8.17. So we assume $n\geqslant 4$. Blow-up x, as usual; let $\mathcal{X}'$ be the new total space and E the exceptional divisor. By Proposition 6.24, three cases may occur: either

(i) E has global normal crossings, i.e. E is the union of a smooth rational normal scroll $X_1 = S(1, d-1)$ of degree d, $2 \leqslant d \leqslant n$, and of $n-d$ disjoint planes $P_1, \ldots, P_{n-d}$, each meeting X_1 along different lines of the same ruling; or

(ii) E is a union of n planes $P_1, \ldots, P_n$ with h Zappatic singular points $x_1, \ldots, x_h$ of type $S_{m_1}, \ldots, S_{m_h}$ such that $3 \leqslant m_i < n$, $i = 1, \ldots, h$, and (6.26) holds; or

(iii) E is a union of n planes with one S_n-point x'.

In case (iii), one has to repeat the process by blowing-up x'. After finitely many steps (cf. e.g. Proposition 3.4.13 in [34]), the exceptional divisor will necessarily be as in cases either (i) or (ii). We remark that no contribution to K^2 comes from the surfaces created in the intermediate steps. Indeed, by using the same notation of the R_n-case in Claim 8.17, if x is a S_n point and if Π_1 is the plane corresponding to the vertex of valence $n-1$ in the associated graph, we have (cf. Figure 18):

$$(K_{P_1'} + \Lambda_1 + A_1 + (n-1)(\Lambda_1 - A_1))^2 = (n-3)^2 - (n-3)^2 = 0,$$
$$(K_{P_i'} + \Lambda_i + A_i + (\Lambda_i - A_i))^2 = 1 - 1 = 0, \quad 2 \leqslant i \leqslant n.$$

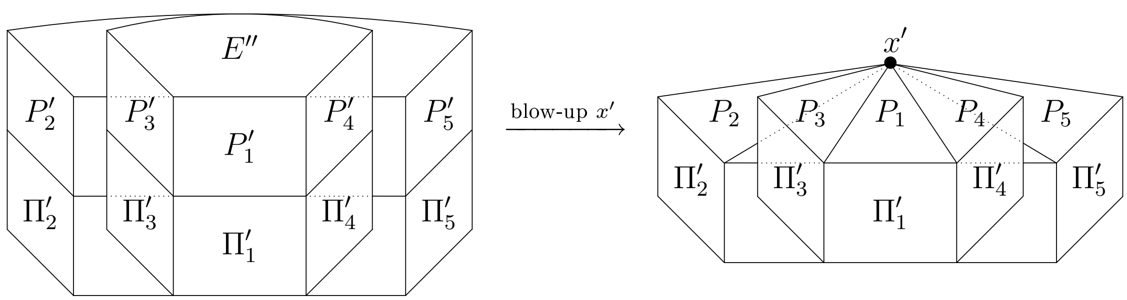


FIGURE 18. Blowing-up a S_5-point x' infinitely near to a S_5-point x

Thus, it suffices to prove the statement for the first two cases (i) and (ii).

Consider the case (i), namely E has global normal crossings. Then $\mathfrak{X}'$ is Gorenstein and we may compute K^2 as in (8.18). The contribution of the blown-up planes $\Pi_1', \ldots, \Pi_n'$ (choosing again the indexes in such a way that Π_1' meets $\Pi_2', \ldots, \Pi_n'$ in a line) is:

$$(K_{X'|\Pi_i'} + E_i)^2 = (w_i - 3)^2 - 1, \qquad i = 2, \ldots, n,$$
$$(K_{X'|\Pi_1'} + E_1)^2 = (w_1 - 3)^2 - (n-3)^2, \tag{8.29}$$

whereas the contribution of E turns out to be:

$$(K_E + \Gamma)^2 = 4 - n. \tag{8.30}$$

Indeed, one finds that:

$$\big((K_E + \Gamma)_{|X_1}\big)^2 = (-A + (n-d-1)F)^2 = d + 4 - 2n,$$
$$\big((K_E + \Gamma)_{|P_i}\big)^2 = 1, \quad i = 1, \ldots, n-d,$$

where A is the linear directrix of X_1 and F is its fibre, therefore (8.30) holds. Summing up, it follows that

$$c_x = n - 4 + (n-1) + (n-3)^2 = (n-2)^2, \tag{8.31}$$

which proves (8.28) in this case (i).

In case (ii), E is not Gorenstein, nonetheless we can compute K^2 since we know (the upper and lower bounds of) the contribution of x_i by induction. We can indeed proceed as in case (ii) of the proof of Claim 8.20, namely, we have to add up three quantities:

- the contribution of $(K_{X'}+\Gamma)^2$, which has been computed in (8.29);
- the contribution to K^2 of E, as if E had only global normal crossings, which is:
$$\Big(K_{P_1}+E_1+\sum_{i=2}^{n}A_i\Big)^2+\sum_{i=2}^{n}(K_{P_i}+E_i+A_i)^2=(n-3)^2+n-1,$$
where Π_1' is the blown-up plane meeting all the other blown-up planes in a line, E_i is the exceptional curve on Π_i' and A_i is the double line intersection of P_1 with P_i;
- the contribution $\sum_{i=1}^{h} c_{x_i}$ of the points x_i, which by induction, is such that:
$$\sum_{i=1}^{h}(m_i-2)^2 \geqslant \sum_{i=1}^{h} c_{x_i} \geqslant \sum_{i=1}^{h}\binom{m_i-1}{2}=\binom{n-1}{2}, \tag{8.32}$$
where the last equality is just (6.26).

Putting all together, one sees that
$$c_x=\sum_{i=1}^{h} c_{x_i},$$
hence (8.32) gives the claimed lower bound, as for the upper bound:
$$c_x\leqslant\sum_{i=1}^{h}(m_i-2)^2=\sum_{i=1}^{h}(m_i-1)(m_i-2)-\sum_{i=1}^{h}(m_i-2)\overset{(*)}{=}$$
$$\overset{(*)}{=}(n-1)(n-2)-\sum_{i=1}^{h}(m_i-2)\leqslant(n-1)(n-2)-(n-2)=(n-2)^2,$$
where the equality $(*)$ follows from (6.26). This completes the proof of Claim 8.27. □

The above Claims 8.23 and 8.27 prove Proposition 8.16 and, so, Theorem 8.1. □

Remark 8.33. Notice that the upper bound $c_x=(n-2)^2$ is attained when for example the exceptional divisor E has global normal crossings (cf. case (i) in Claim 8.27). The lower bound $c_x=\binom{n-1}{2}$ can be attained if the exceptional divisor E consists of n planes forming $\binom{n-1}{2}$ points of type $S_3=R_3$.

Contrary to what happens for the R_n-points, not all the values between the upper and the lower bound are realized by c_x, for a S_n-point x. Indeed they are not even combinatorially possible. For example, consider the case of a S_6-point x: the bounds in (8.28) say that $10\leqslant c_x\leqslant 16$. If the exceptional divisor E has global normal crossings, then $c_x=16$. Otherwise E is a union of planes and has the following Zappatic singularities: $q_5\leqslant 1$ points of type S_5, $q_4\leqslant 3-2q_5$ points of type S_5 and $q_3=10-3q_4-6q_5$ points of type $S_3=R_3$. It follows that
$$c_x\leqslant q_3+4q_4+9q_5=10+q_4+3q_5\leqslant 13+q_5\leqslant 14.$$
Therefore the case $c_x=15$ cannot occur if x is a S_6-point.

9. The genus of the fibres of degenerations of surfaces to Zappatic ones

In this section we want to investigate on the behaviour of the geometric genus of the smooth fibres of a degeneration of surfaces to a good Zappatic one, in terms of the ω-genus of the central fibre.

By recalling Definition 5.1, the geometric genus of the general fibre of a semistable degeneration of surfaces can be computed via the *Clemens-Schmid exact sequence*, cf. [43]. Clemens-Schmid result implies the following:

Theorem 9.1. *Let $X = \bigcup_{i=1}^{v} X_i$ be the central fibre of a semistable degeneration of surfaces $\mathfrak{X} \to \Delta$. Let G_X be the graph associated to X and Φ_X be the map introduced in Definition 4.1. Then, for $t \neq 0$, one has:*

$$p_g(\mathfrak{X}_t) = h^2(G_X, \mathbb{C}) + \sum_{i=1}^{v} p_g(X_i) + \dim(\operatorname{coker}(\Phi_X)). \tag{9.2}$$

Then Theorem 9.1 and our Theorem 4.3 imply the following:

Corollary 9.3. *Let $\mathfrak{X} \to \Delta$ be a semistable degeneration of surfaces, so that its central fibre $X = \mathfrak{X}_0$ is a good Zappatic surface with only E_3-points as Zappatic singularities. Then, for any $t \neq 0$, one has:*

$$p_g(\mathfrak{X}_t) = p_\omega(X).$$

Remark 9.4. Let $\mathfrak{X} \to \Delta$ be a degeneration of surfaces with central fibre X. Consider the dualizing sheaf $\omega_{\mathfrak{X}}$ of $\mathfrak{X}$. By general properties of dualizing sheaves, one knows that $\omega_{\mathfrak{X}}$ is torsion-free as an $\mathcal{O}_{\mathfrak{X}}$-module. Since one has the injection $\mathcal{O}_\Delta \hookrightarrow \mathcal{O}_{\mathfrak{X}}$, then $\omega_{\mathfrak{X}}$ is torsion-free over Δ. Since Δ is the spectrum of a DVR, then $\omega_{\mathfrak{X}}$ is free and therefore flat over Δ. By semi-continuity, this implies that, for $t \neq 0$, $p_g(\mathfrak{X}_t) \leqslant p_\omega(X)$. The above corollary shows that equality holds for semistable degenerations of surfaces.

Consider, from now on, a degeneration $\pi : \mathfrak{X} \to \Delta$ of surfaces with good Zappatic central fibre $X = \mathfrak{X}_0$. Our main purpose in this section is to prove Proposition 9.7, where we show that the ω-genus of the central fibre of a semistable reduction $\tilde{\pi} : \tilde{\mathfrak{X}} \to \Delta$ of π equals the ω-genus of X. As a consequence we will have that the ω–genus of the fibres of $\pi : \mathfrak{X} \to \Delta$ is constant (see Theorem 9.9 below), exactly as it happens in the normal crossings case, as we saw in Corollary 9.3.

In order to prove Proposition 9.7, we make use of Proposition 7.4. Indeed, let $X = \bigcup_{i=1}^{v} X_i$ be the central fibre of the original degeneration and let $\bar{X}_{\rm red} = \bigcup_{i=1}^{w} \bar{X}_i$ be the support of the central fibre $\bar{X}$ of its normal crossing reduction obtained as in Proposition 7.4, where $w \geqslant v$. Next we describe the relation between the graph G associated to X and the one $\bar{G}$ associated to $\bar{X}_{\rm red}$. By the proof of Proposition 7.4, one has that G is a subgraph of $\bar{G}$ and we may assume that $\bar{X}_i$ is birational to X_i, $i = 1, \ldots, v$.

Proposition 9.5. *(cf.* [10, Proposition 4.10]*) In the above situation, one has:*

(i) $p_g(\bar{X}_i) = 0$, *$i = v+1, \ldots, w$;*
(ii) $\dim(\operatorname{coker}(\Phi_{\bar{X}_{\rm red}})) = \dim(\operatorname{coker}(\Phi_X))$*;*
(iii) *the graphs G and $\bar{G}$ have the same Betti numbers.*

Proof. Following the discussion of Gorenstein reduction algorithm 7.2 and of each Step of the normal crossing reduction algorithm 7.3, one sees that each new component $\bar{X}_i$, $i =$

$v+1,\dots,w$, of the central fibre is an exceptional divisor of a blow-up, which is either a rational or a ruled surface. This proves (i).

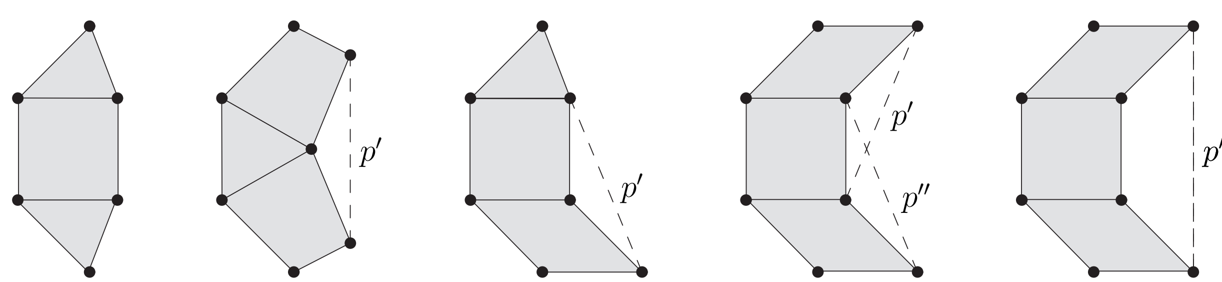
For $i=1,\dots,v$, the birational morphism $\bar\sigma : \bar{\mathfrak{X}} \to \mathfrak{X}$ determines a birational morphism $\bar X_i \to X_i$ which is the composition of blow-ups at smooth points of X_i. In order to prove (ii), we notice that in algorithms 7.2 and 7.3, we have added rational double curves (which do not contribute to the cokernel), new rational components (which also do not contribute to the cokernel), and irrational ruled surfaces, which are only created by blowing-up irrational double curves. Focusing on single such irrational double curve, one sees that it is replaced by a certain number h of irrational ruled surfaces, and by $h+1$ new double curves. The map on the H^1 level is an isomorphism between the new surfaces and the new curves. Hence there is no change in the dimension of the cokernel. This concludes the proof of (ii).

In order to prove (iii), let us see what happens at algorithm 7.2 and at each step of algorithm 7.3.

In algorithm 7.2, one blows-up R_n- and S_n-points of $X=\mathfrak{X}_0$. An example will illustrate the key features of the analysis. Let p be a R_4-point of X. After blowing-up $\mathfrak{X}$ at p, there are five different possible configurations of the exceptional divisor E (cf. the proof of Claim 8.20):

(i) E is the union of two quadrics with normal crossings;
(ii) E is the union of a quadric and two planes having a R_3-point p', and the quadric is in the middle;
(iii) E is the union of a quadric and two planes having a R_3-point p', and one of the planes is in the middle;
(iv) E is the union of four planes having two R_3-points p', p'';
(v) E is the union of four planes having a R_4-point p'.

The corresponding associated graphs are illustrated in Figure 19, where the proper transforms of the four components of X concurring at p are the left-hand-side vertices in each graph. As the pictures show, G is a deformation retract of the new associated graph (considered as CW-complexes).



Case (i) Case (ii) Case (iii) Case (iv) Case (v)

FIGURE 19. After blowing-up a R_4-point p, there are five possibilities

Generally, if one blows-up a R_n- [resp. S_n-] point p, in the associated graph to X one builds new 3- and 4- faces (triangles and quadrangles) over the original chain of length n [resp. fork with $n-1$ teeth] corresponding to the n components of X concurring at p. Therefore it is always the case that G is a deformation retract of the new associated graph.

From this point on there are no more R_n or S_n points ever appearing in the configuration. However it may happen that at intermediate steps of the algorithm, we do not have strict normal crossings nor Zappatic singularities. If this happens, we still consider the usual associated graph to the configuration, namely a vertex for each component, an edge for each connected component of an intersection between components, and faces for intersections of three or more components.

Consider Step 1 of algorithm 7.3. Each blow-up of an E_n-point, where the total space has multiplicity n, has the effect of adding new vertices in the interior of the corresponding n-face and of adding new edges which subdivide the n-face. This does not modify the Betti numbers of the associated graph.

In Step 2 of algorithm 7.3, the blow-up along a double curve determines a subdivision of the edge corresponding to the double curve and a subdivision of the faces adjacent on that edge.

In Step 3 of algorithm 7.3, the blow-ups at double points of types (a) and (b) add trees adjacent only to a vertex or an edge, and again this does not modify the topological properties of the graph.

Resolving a double point of type (c), one first subdivides the original triangle of vertices v_1, v_2, v_3 in three triangles; then, setting v_0 the new vertex, one adds another vertex v_0' above v_0 and three triangles of vertices v_0, v_0', v_i, respectively for $i = 1, 2, 3$. Clearly the resulting graph retracts back to a subdivision of the original one.

For a double point of type (d), one subdivides the original quadrangle either in four triangles, if the exceptional divisor E of the blow-up is a smooth quadric, or in two triangles and two quadrangles as in Figure 20, if E is the union of two planes.

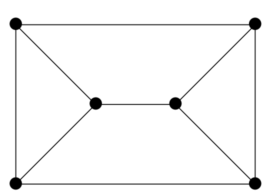


FIGURE 20. Subdivision of a quadrangle in type (d), case (ii)

For a double point of type (e), one subdivides the original triangle either in three triangles, if the exceptional divisor E of the blow-up is irreducible, or in a triangle and two quadrangles as in Figure 21, if E is reducible.

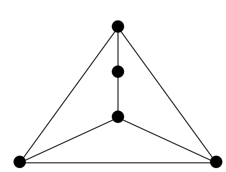


FIGURE 21. Subdivision of a triangle in type (e), case (iii)

In all cases, one sees that these modifications, coming from the resolution of double points of type (c), (d) and (e), do not change the Betti numbers of the associated graph.

Finally, the blow-ups of Steps 4 and 5 add trees adjacent to a vertex or an edge and again do not modify the Betti number of the associated graph. □

We are interested not only in $\bar{X}_{\rm red}$ but in $\bar{X}$ itself. For each component i, let μ_i be the multiplicity of $\bar{X}_i$ in $\bar{X}$. For the analysis of the semistable reduction, we must understand rather precisely the components of multiplicity larger than one.

Corollary 9.6. *(cf.* [10, Corollary 4.11]*) Set $\bar{C}_{ij} = \bar{X}_i \cap \bar{X}_j$ if $\bar{X}_i$ and $\bar{X}_j$ meet along a curve, or $\bar{C}_{ij} = \emptyset$ otherwise. If $\mu_i > 1$, one has the following possibilities:*

(i) *$\bar{X}_i$ is a generically ruled surface and the curve $\sum_{j\neq i} \mu_j \bar{C}_{ij}$ is generically supported on a bisection of the ruling.*

(ii) *There is a birational morphism $\sigma : \bar{X}_i \to \mathbb{P}^2$ such that the curve $\sum_{j\neq i} \mu_j \bar{C}_{ij}$ maps to four distinct lines.*

(iii) *$\mu_i = 4$ and $\bar{X}_i$ is a smooth quadric; the curve $\sum_{j\neq i} \mu_j \bar{C}_{ij}$ consists of two (multiplicity one) fibres in one ruling and one double fibre from the other ruling.*

(iv) *$\bar{X}_i$ is a smooth quadric and the curve $\sum_{j\neq i} \mu_j \bar{C}_{ij}$ is linearly equivalent to $\mu_i H$, where H is a plane section of $\bar{X}_i$.*

(v) *There is a birational morphism $\sigma : \bar{X}_i \to \mathbb{P}^2$ such that the curve $\sum_{j\neq i} \mu_j \bar{C}_{ij}$ is the total transform via σ of a plane curve of degree μ_i supported on two distinct lines.*

(vi) *$\bar{X}_i$ is a Hirzebruch surface $\mathbb{F}_2$ and the curve $\sum_{j\neq i} \mu_j \bar{C}_{ij}$ is of the form $\mu_i(H + A)$, where A is the (-2)-curve and H is a section of self-intersection 2.*

Proof. Following the the Gorenstein reduction algorithm 7.2 and the steps of the normal crossing reduction algorithm 7.3, one sees that multiple components are not created either in algorithm 7.2 or in Step 1 of algorithm 7.3. It is possible that a multiple component may be created in Step 2, by blowing-up a double curve of $X_{\rm red}$ which is the intersection of two components that have multiplicity. This will create a multiple ruled surface whose double curve is a bisection, giving case (i).

Multiple components of the central fibre $\bar{X}$ may arise also in Step 3 when one blows-up double points of types (c), (d) and (e). In case (c), two types of multiple components appear. The first is a plane blown-up at three collinear points, with multiplicity two; the double curve consists of the collinearity line, three other general lines, and the three exceptional divisors counted with multiplicity four; this is case (ii). The other type of multiple component is a quadric with multiplicity four, giving case (iii). This analysis follows from the remark we did at the end of Step 4, where we showed that the three surfaces coming together to form this singularity of type (c) each have multiplicity one.

Let $p = X_1 \cap X_2 \cap X_3 \cap X_4$ be a point of type (d), where $X_1, \ldots, X_4$ are irreducible components of $X_{\rm red}$. One may choose the numbering on the four components such that $X_1 \cup X_2$ and $X_3 \cup X_4$ are local complete intersections of $\mathfrak{X}$ at p, and moreover the multiplicities satisfy $\mu_1 = \mu_2$ and $\mu_3 = \mu_4$. (This is clear at the start, when all multiplicities are one; and from that point on one proceeds inductively.) Then the exceptional divisor E appears in the new central fibre with multiplicity $\mu_1 + \mu_3 = \mu_2 + \mu_4$. Recall that if E is a smooth quadric, the resolution process stops, and we have case (iv) above; while if E is the union of two planes, then both planes appear with multiplicity $\mu_1 + \mu_3$ and we go on inductively; this gives case (v).

Let now $p = X_1 \cap X_2 \cap X_3$ be a point of type (e). As noted above, $X_2 \cup X_3$ and X_1 are local complete intersections of $\mathfrak{X}$ at p. As above, one may assume that the multiplicities satisfy

$\mu_2 = \mu_3$. Then the exceptional divisor E appears in the new central fibre with multiplicity $\mu_1 + \mu_2 = \mu_1 + \mu_3$. If E is a smooth quadric, the resolution process stops, giving case (iv) again. If E is a quadric cone, then we proceed to blow-up the vertex of the cone, and therefore the proper transform of E in the final central fibre will be a Hirzebruch surface $\mathbb{F}_2$, which gives the final case (vi). Finally if E is a pair of planes, each plane gives rise to a component in case (v). □

Now, by recalling Remark 5.2, we are able to prove the main result of this section:

Proposition 9.7. *(cf.* [10, Proposition 4.12]*) Let $\pi : \mathfrak{X} \to \Delta$ be a degeneration of surfaces with good Zappatic central fibre $X = \mathfrak{X}_0 = \bigcup_{i=1}^{v} X_i$. Let $\bar{\pi} : \bar{\mathfrak{X}} \to \Delta$ be the normal crossing reduction of π given by algorithms 7.2 and 7.3 and let $\tilde{\pi} : \tilde{\mathfrak{X}} \to \Delta$ be the semistable reduction of $\bar{\pi}$ obtained by following the process described in Chapter II of* [33]*. Then:*

$$p_\omega(\tilde{\mathfrak{X}}_0) = p_\omega(X). \tag{9.8}$$

Proof. Let $\bar{X} = \bar{\mathfrak{X}}_0 = \sum_{i=1}^{w} \mu_i \bar{X}_i$ be the central fibre of the normal crossing reduction $\bar{\pi}$. One has $v \leqslant w$ and we may assume that $\mu_i = 1$ for $1 \leqslant i \leqslant v$, and that these first v components are birational to the original components of X. The surface $\bar{X}$ is a toroidal embedding in $\bar{\mathfrak{X}}$, in the sense of Definition 1, p. 54 of [33]. To any such a toroidal embedding one can associate a compact polyhedral complex $\bar{\Gamma}$ with integral structure as shown in [33], pp. 71 and 94. In our present situation, the complex $\bar{\Gamma}$ is exactly the associated graph $\bar{G}$. The integral structure is recorded by the multiplicities of the components.

By [33], p. 107, there exists a semistable reduction $\tilde{\mathfrak{X}} \to \Delta$ as in Diagram 5.3, where the base change $\beta(t) = t^m$ is such that m is a common multiple of $\mu_1, \dots, \mu_w$. Notice that $\tilde{\mathfrak{X}}$ is again a toroidal embedding of the central fibre $\tilde{X} = \tilde{\mathfrak{X}}_0$. Denote by $\tilde{G}$ the associated graph to $\tilde{X}$. Again by [33], p. 107, one has that the corresponding polyhedron $\tilde{\Gamma}$ is a subdivision of $\bar{\Gamma}$, in the sense of the definition at p. 111 of [33]. This implies that the CW-complexes $\tilde{G}$ and $\bar{G}$ are homeomorphic. In particular they have the same homology.

Now the central fibre $\tilde{X} = \tilde{\mathfrak{X}}_0 = \bigcup_{i=1}^{u} \tilde{X}_i$ is reduced, with global normal crossings. One has that $u \geqslant w$ and, by taking into account the base change, one may assume that, for $i = 1, \dots, w$, $\tilde{X}_i$ is birational to the μ_i-tuple cover of $\bar{X}_i$, branched along $\sum_{j \neq i} \mu_j \bar{C}_{ij}$.

Let us first consider components with $\mu_i = 1$. These include the first v components $\tilde{X}_i$, $i = 1, \dots, v$, which correspond to the original components of X. For these components we have $p_g(\tilde{X}_i) = p_g(\bar{X}_i) = p_g(X_i)$, $i = 1, \dots, v$. There also may be components with $\mu_i = 1$ which were introduced in the normal crossing reduction algorithm. We have seen in Proposition 9.5 that all such components have $p_g = 0$. Finally there may be components with $\mu_i = 1$ with $i > w$ which have been introduced in the semistable reduction process. These new surfaces are of two types: they may correspond either to

(a) vertices of $\tilde{G}$ which lie on an edge η of $\bar{G}$; or to
(b) vertices of $\tilde{G}$ which lie in the interior of a triangular face of $\bar{G}$.

We recall that the birational morphism $\tilde{\mathfrak{X}} \to \mathfrak{X}_\beta$ as in Diagram 5.3 is the blow-up of a suitable sheaf of ideals, cf. p. 107 of [33].

Let $\tilde{X}_j$ be a surface of type (a). This is an exceptional divisor of such a blow-up with support on the double curve γ of $\bar{X}$ corresponding to the edge η. Then $\tilde{X}_j$ maps to γ with fibres which are rational by the toric nature of the singularity along γ.

Suppose that $\tilde{X}_j$ is of type (b). Then $\tilde{X}_j$ is an exceptional divisor appearing in the toric resolution of a toric singular point. Therefore $\tilde{X}_j$ is rational and moreover it meets the other components along rational curves (cf., e.g., Section 2.6 in [22]).

Therefore all of these components are rational or ruled, and hence also have $p_g = 0$.

Now let us consider the case $\mu_i > 1$. In this case $\tilde{X}_i$ is a μ_i-cover of the surface $\bar{X}_i$, and such surfaces were classified in the previous corollary, along with the double curves which give the branch locus of the covering. In each case the cover is easily seen to be rational or ruled. Hence also for these surfaces one has $p_g = 0$.

Since we have shown that the homology of the graphs are the same, and we have controlled the p_g of the components properly, the only thing left to prove is that $\dim(\mathrm{coker}(\Phi_X)) = \dim(\mathrm{coker}(\Phi_{\tilde{X}}))$.

We have already seen that $\dim(\mathrm{coker}(\Phi_{\bar{X}_{\mathrm{red}}})) = \dim(\mathrm{coker}(\Phi_X))$ in Proposition 9.5. The argument here is similar; it suffices to show that the extra components $\tilde{X}_{v+1}, \ldots, \tilde{X}_u$ do not contribute to $\dim(\mathrm{coker}(\Phi_{\tilde{X}}))$. These surfaces are either rational or ruled over a curve γ. In the rational case, by the proof of Proposition 7.3 and by the above considerations about toric resolution of singularities, they meet the other components of $\tilde{X}$ along rational curves. Hence they do not contribute to $\dim(\mathrm{coker}\,\Phi_{\tilde{X}})$.

In the ruled case, $\tilde{X}_j$ is a scroll over γ and, by the description of the resolution process, $\tilde{X}_j$ meets the other components of $\tilde{X}$ along curves which are either rational or isomorphic to γ. The same argument as in Proposition 9.5 shows that the cokernel is unchanged in this case.

Thus the proof is concluded by Theorem 4.3. $\square$

As a direct consequence, we have the following:

Theorem 9.9. *(cf.* [10, Theorem 4.14]*) Let $\pi : \mathcal{X} \to \Delta$ be a degeneration of surfaces with good Zappatic central fibre $X = \mathcal{X}_0$. Then, for any $t \neq 0$, one has:*

$$p_g(\mathcal{X}_t) = p_\omega(X).$$

Proof. Just consider the semistable reduction $\tilde{\pi} : \tilde{\mathcal{X}} \to \Delta$ as we did before. One clearly has that $p_g(\mathcal{X}_t) = p_g(\tilde{\mathcal{X}}_t)$ for $t \neq 0$. Theorem 9.1 then implies that $p_g(\tilde{\mathcal{X}}_t) = p_\omega(\tilde{\mathcal{X}}_0)$ and finally Proposition 9.7 concludes that $p_\omega(\tilde{\mathcal{X}}_0) = p_\omega(X)$. $\square$

10. The Multiple Point Formula

The aim of this section is to prove a fundamental inequality, which involves the Zappatic singularities of a given good Zappatic surface X (see Theorem 10.2), under the hypothesis that X is the central fibre of a good Zappatic degeneration as in Definition 5.4. This inequality can be viewed as an extension of the well-known Triple Point Formula (see Lemma 10.7 and cf. [20]), which holds only for semistable degenerations. As corollaries, we will obtain, among other things, the main result contained in Zappa's paper [60] (cf. Section 11).

Let us introduce some notation.

Notation 10.1. Let X be a good Zappatic surface. We denote by:

- $\gamma = X_1 \cap X_2$ the intersection of two irreducible components X_1, X_2 of X;
- F_γ the divisor on γ consisting of the E_3-points of X along γ;
- $f_n(\gamma)$ the number of E_n-points of X along γ; in particular, $f_3(\gamma) = \deg(F_\gamma)$;
- $r_n(\gamma)$ the number of R_n-points of X along γ;

- $s_n(\gamma)$ the number of S_n-points of X along γ;
- $\rho_n(\gamma) := r_n(\gamma) + s_n(\gamma)$, for $n \geqslant 4$, and $\rho_3(\gamma) = r_3(\gamma)$.

If X is the central fibre of a good Zappatic degeneration $\mathcal{X} \to \Delta$, we denote by:

- D_γ the divisor of γ consisting of the double points of $\mathcal{X}$ along γ off the Zappatic singularities of X;
- $d_\gamma = \deg(D_\gamma)$;
- $d_{\mathcal{X}}$ the total number of double points of $\mathcal{X}$ off the Zappatic singularities of X.

The main result of this section is the following (cf. [9, Theorem 7.2]):

Theorem 10.2 (Multiple Point Formula)**.** *Let X be a surface which is the central fibre of a good Zappatic degeneration $\mathcal{X} \to \Delta$. Let $\gamma = X_1 \cap X_2$ be the intersection of two irreducible components X_1, X_2 of X. Then*

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) - r_3(\gamma) - \sum_{n\geqslant 4}(\rho_n(\gamma) + f_n(\gamma)) \geqslant d_\gamma \geqslant 0. \tag{10.3}$$

In the planar case, one has:

Corollary 10.4. *Let X be a surface which is the central fibre of a good, planar Zappatic degeneration $\mathcal{X} \to \Delta$. Let γ be a double line of X. Then*

$$2 + f_3(\gamma) - r_3(\gamma) - \sum_{n\geqslant 4}(\rho_n(\gamma) + f_n(\gamma)) \geqslant d_\gamma \geqslant 0. \tag{10.5}$$

Therefore:

$$2e + 3f_3 - 2r_3 - \sum_{n\geqslant 4} n f_n - \sum_{n\geqslant 4}(n-1)\rho_n \geqslant d_{\mathcal{X}} \geqslant 0. \tag{10.6}$$

As for Theorem 8.1, the proof of Theorem 10.2 will be done in several steps, the first of which is the classical:

Lemma 10.7 (Triple Point Formula)**.** *Let X be a good Zappatic surface with global normal crossings, which is the central fibre of a good Zappatic degeneration with smooth total space $\mathcal{X}$. Let $\gamma = X_1 \cap X_2$, where X_1 and X_2 are irreducible components of X. Then:*

$$\mathcal{N}_{\gamma|X_1} \otimes \mathcal{N}_{\gamma|X_2} \otimes \mathcal{O}_\gamma(F_\gamma) \cong \mathcal{O}_\gamma. \tag{10.8}$$

In particular,

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) = 0. \tag{10.9}$$

Proof. By Definition 5.4, since the total space $\mathcal{X}$ is assumed to be smooth, the good Zappatic degeneration $\mathcal{X} \to \Delta$ is semistable. Let $X = \bigcup_{i=1}^{v} X_i$. Since X is a Cartier divisor in $\mathcal{X}$ which is a fibre of the morphism $\mathcal{X} \to \Delta$, then $\mathcal{O}_X(X) \cong \mathcal{O}_X$. Tensoring by $\mathcal{O}_\gamma$ gives $\mathcal{O}_\gamma(X) \cong \mathcal{O}_\gamma$. Thus,

$$\mathcal{O}_\gamma \cong \mathcal{O}_\gamma(X_1) \otimes \mathcal{O}_\gamma(X_2) \otimes \mathcal{O}_\gamma(Y), \tag{10.10}$$

where $Y = \cup_{i=3}^{v} X_i$. One concludes by observing that in (10.10) one has $\mathcal{O}_\gamma(X_i) \cong \mathcal{N}_{\gamma|X_{3-i}}$, $1 \leqslant i \leqslant 2$, and $\mathcal{O}_\gamma(Y) \cong \mathcal{O}_\gamma(F_\gamma)$. $\square$

It is useful to consider the following slightly more general situation. Let X be a union of surfaces such that $X_{\rm red}$ is a good Zappatic surface with global normal crossings. Then $X_{\rm red} = \cup_{i=1}^{v} X_i$ and let m_i be the multiplicity of X_i in X, $i = 1, \ldots, v$. Let $\gamma = X_1 \cap X_2$ be the intersection of two irreducible components of X. For every point p of γ, we define the weight $w(p)$ of p as the multiplicity m_i of the component X_i such that $p \in \gamma \cap X_i$.

Of course $w(p) \neq 0$ only for E_3-points of $X_{\rm red}$ on γ. Then we define the divisor F_γ on γ as

$$F_\gamma := \sum_p w(p) p.$$

The same proof of Lemma 10.7 shows the following:

Lemma 10.11 (Generalized Triple Point Formula). *Let X be a surface such that $X_{\rm red} = \cup_i X_i$ is a good Zappatic surface with global normal crossings. Let m_i be the multiplicity of X_i in X. Assume that X is the central fibre of a degeneration $\mathcal{X} \to \Delta$ with smooth total space $\mathcal{X}$. Let $\gamma = X_1 \cap X_2$, where X_1 and X_2 are irreducible components of $X_{\rm red}$. Then:*

$$\mathcal{N}_{\gamma|X_1}^{\otimes m_2} \otimes \mathcal{N}_{\gamma|X_2}^{\otimes m_1} \otimes \mathcal{O}_\gamma(F_\gamma) \cong \mathcal{O}_\gamma. \tag{10.12}$$

In particular,

$$m_2 \deg(\mathcal{N}_{\gamma|X_1}) + m_1 \deg(\mathcal{N}_{\gamma|X_2}) + \deg(F_\gamma) = 0. \tag{10.13}$$

The second step is given by the following result (cf. [9, Proposition 7.14]):

Proposition 10.14. *Let X be a good Zappatic surface with global normal crossings, which is the central fibre of a good Zappatic degeneration $\mathcal{X} \to \Delta$. Let $\gamma = X_1 \cap X_2$, where X_1 and X_2 are irreducible components of X. Then:*

$$\mathcal{N}_{\gamma|X_1} \otimes \mathcal{N}_{\gamma|X_2} \otimes \mathcal{O}_\gamma(F_\gamma) \cong \mathcal{O}_\gamma(D_\gamma). \tag{10.15}$$

In particular,

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) = d_\gamma. \tag{10.16}$$

Proof. By the very definition of good Zappatic degeneration, the total space $\mathcal{X}$ is smooth except for ordinary double points along the double locus of X, which are not the E_3-points of X. We can modify the total space $\mathcal{X}$ and make it smooth by blowing-up its double points.

Since the computations are of local nature, we can focus on the case of $\mathcal{X}$ having only one double point p on γ. We blow-up the point p in $\mathcal{X}$ to get a new total space $\mathcal{X}'$, which is smooth. Notice that, according to our hypotheses, the exceptional divisor $E := E_{\mathcal{X},p} = \mathbb{P}(T_{\mathcal{X},p})$ is isomorphic to a smooth quadric in $\mathbb{P}^3$ (see Figure 22).

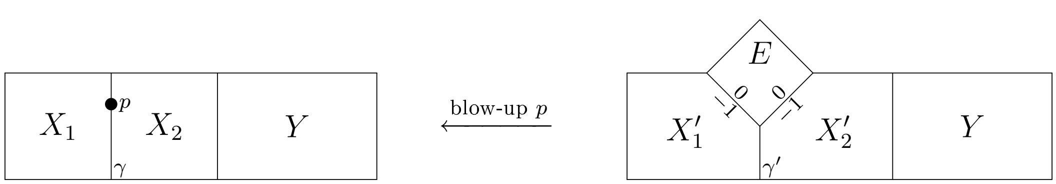


FIGURE 22. Blowing-up an ordinary double point of $\mathcal{X}$

The proper trasform of X is:

$$X' = X'_1 + X'_2 + Y$$

where X_1', X_2' are the proper transforms of X_1, X_2, respectively. Let γ' be the intersection of X_1' and X_2', which is clearly isomorphic to γ. Let p_1 be the intersection of γ' with E.

Since $\mathcal{X}'$ is smooth, we can apply Lemma 10.11 to γ'. Therefore, by (10.12), we get

$$\mathcal{O}_{\gamma'} \cong \mathcal{N}_{\gamma'|X_1'} \otimes \mathcal{N}_{\gamma'|X_2'} \otimes \mathcal{O}_{\gamma'}(F_{\gamma'}).$$

In the isomorphism between γ' and γ, one has:

$$\mathcal{O}_{\gamma'}(F_{\gamma'} - p_1) \cong \mathcal{O}_{\gamma}(F_{\gamma}), \quad \mathcal{N}_{\gamma'|X_i'} \cong \mathcal{N}_{\gamma|X_i} \otimes \mathcal{O}_{\gamma}(-p), \quad 1 \leqslant i \leqslant 2.$$

Putting all this together, one has the result. $\square$

Taking into account Lemma 10.11, the same proof of Proposition 10.14 gives the following result:

Corollary 10.17. *(cf.* [9, Corollary 7.17]*) Let X be a surface such that $X_{\mathrm{red}} = \cup_i X_i$ is a good Zappatic surface with global normal crossings. Let m_i be the multiplicity of X_i in X. Assume that X is the central fibre of a degeneration $\mathcal{X} \to \Delta$ with total space $\mathcal{X}$ having at most ordinary double points outside the Zappatic singularities of X_{red}.*

Let $\gamma = X_1 \cap X_2$, where X_1 and X_2 are irreducible components of X_{red}. Then:

$$\mathcal{N}_{\gamma|X_1}^{\otimes m_2} \otimes \mathcal{N}_{\gamma|X_2}^{\otimes m_1} \otimes \mathcal{O}_{\gamma}(F_{\gamma}) \cong \mathcal{O}_{\gamma}(D_{\gamma})^{\otimes (m_1+m_2)}. \tag{10.18}$$

In particular,

$$m_2 \deg(\mathcal{N}_{\gamma|X_1}) + m_1 \deg(\mathcal{N}_{\gamma|X_2}) + \deg(F_{\gamma}) = (m_1 + m_2) d_{\gamma}. \tag{10.19}$$

Now we can come to the:

Proof of Theorem 10.2. Recall that, by Definition 5.4 of Zappatic degenerations, the total space $\mathcal{X}$ has only isolated singularities. We want to apply Corollary 10.17 after having resolved the singularities of the total space $\mathcal{X}$ at the Zappatic singularities of the central fibre X, i.e. at the R_n-points of X, for $n \geqslant 3$, and at the E_n- and S_n-points of X, for $n \geqslant 4$.

Now we briefly describe the resolution process, which will become even clearer in the second part of the proof, when we will enter into the details of the proof of Formula (10.3).

Following the blowing-up process of Algorithm 7.2 and of Proposition 7.3 - (1) at the R_n- and S_n-points of the central fibre X, as described in details in Section 8, one gets a degeneration such that the total space is Gorenstein, with isolated singularities, and the central fibre is a Zappatic surface with only E_n-points.

The degeneration will not be Zappatic, if the double points of the total space occurring along the double curves, off the Zappatic singularities, are not ordinary. According to our hypotheses, this cannot happen along the proper transform of the double curves of the original central fibre. All these non-ordinary double points can be resolved with finitely many subsequent blow-ups and they will play no role in the computation of Formula (10.3).

Recall that the total space $\mathcal{X}$ is smooth at the E_3-points of the central fibre, whereas $\mathcal{X}$ has multiplicity either 2 or 4 at an E_4-point of X. Thus, we can consider only E_n-points $p \in X$, for $n \geqslant 4$.

By Proposition 6.17, p is a quasi-minimal singularity for $\mathcal{X}$, unless $n = 4$ and $\mathrm{mult}_p(\mathcal{X}) = 2$. In the latter case, this singularity is resolved by a sequence of blowing-ups at isolated double points.

Assume now that p is a quasi-minimal singularity for $\mathcal{X}$. Let us blow-up $\mathcal{X}$ at p and let E' be the exceptional divisor. Since a hyperplane section of E' is C_{E_n}, the possible configurations of E' are described in Proposition 6.24, (iii).

If E' is irreducible, that is case (iii.a) of Proposition 6.24, then E' has at most isolated rational double points, where the new total space is either smooth or it has a double point. This can be resolved by finitely many blowing-ups at analogous double points.

Suppose we are in case (iii.b) of Proposition 6.24. If E' has global normal crossings, then the desingularization process proceeds exactly as before.

If E' does not have global normal crossings then, either E' has a component which is a quadric cone or the two components of E' meet along a singular conic. In the former case, the new total space has a double point at the vertex of the cone. In the latter case, the total space is either smooth or it has an isolated double point at the singular point of the conic. In either cases, one resolves the singularities by a sequence of blowing-ups as before.

Suppose finally we are in case (iii.c) of Proposition 6.24, i.e. the new central fibre is a Zappatic surface with one point p' of type E_m, with $m \leqslant n$. Then we can proceed by induction on n. Note that if an exceptional divisor has an E_3-point p'', then p'' is either a smooth, or a double, or a triple point for the total space. In the latter two cases, we go on by blowing-up p''. After finitely many blow-ups (by Definition 5.4, cf. Proposition 3.4.13 in [34]), we get a central fibre which might be non-reduced, but its support has only global normal crossings, and the total space has at most ordinary double points off the E_3-points of the reduced part of the central fibre.

Now we are in position to apply Corollary 10.17. In order to do this, we have to understand the relations between the invariants of a double curve of the original Zappatic surface X and the invariants appearing in Formula (10.19) for the double curve of the strict transform of X.

Since all the computations are of local nature, we may assume that X has a single Zappatic singularity p, which is not an E_3-point. We will prove the theorem in this case. The general formula will follow by iterating these considerations for each Zappatic singularity of $\mathcal{X}$.

Let X_1, X_2 be irreducible components of X containing p and let γ be their intersection. As we saw in the above resolution process, we blow-up $\mathcal{X}$ at p. We obtain a new total space $\mathcal{X}'$, with the exceptional divisor $E' := E_{\mathcal{X},p} = \mathbb{P}(T_{\mathcal{X},p})$ and the proper transform X_1', X_2' of X_1, X_2. Let γ' be the intersection of X_1', X_2'. We remark that $\gamma' \cong \gamma$ (see Figure 23).

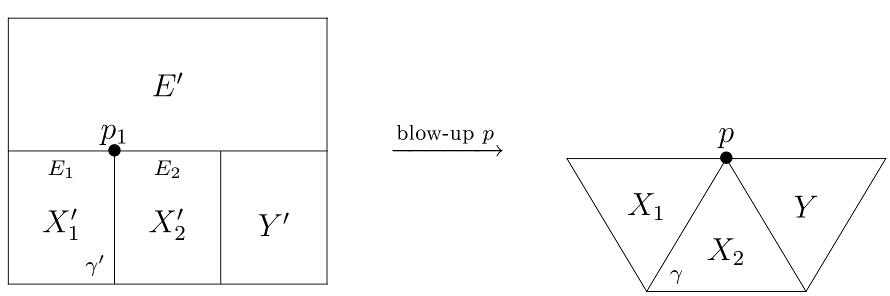


FIGURE 23. Blowing-up $\mathcal{X}$ at p

Notice that $\mathcal{X}'$ might have Zappatic singularities off γ'. These will not affect our considerations. Therefore, we can assume that there are no singularities of $\mathcal{X}'$ of this sort. Thus, the only point of $\mathcal{X}'$ we have to take care of is $p_1 := E' \cap \gamma'$.

If p_1 is smooth for E', then it must be smooth also for $\mathcal{X}'$. Moreover, if p_1 is singular for E', then p_1 is a double point of E' as it follows from the above resolution process and from Proposition 6.24. Therefore, p_1 is at most double also for $\mathcal{X}'$; since p_1 is a quasi-minimal, Gorenstein singularity of multiplicity 4 for the central fibre of $\mathcal{X}'$, then p_1 is a double point of $\mathcal{X}'$ by Proposition 6.17.

Thus there are two cases to be considered: either

(i) p_1 is smooth for both E' and $\mathcal{X}'$, or

(ii) p_1 is a double point for both E' and $\mathcal{X}'$.

In case (i), the central fibre of $\mathcal{X}'$ is $\mathcal{X}'_0 = X'_1 \cup X'_2 \cup Y' \cup E'$ and we are in position to use the enumerative information (10.16) from Proposition 10.14 which reads:

$$\deg(\mathcal{N}_{\gamma'|X'_1}) + \deg(\mathcal{N}_{\gamma'|X'_2}) + f_3(\gamma') = d_{\gamma'}.$$

Observe that $f_3(\gamma')$ is the number of E_3-points of the central fibre $\mathcal{X}'_0$ of $\mathcal{X}'$ along γ', therefore

$$f_3(\gamma') = f_3(\gamma) + 1.$$

On the other hand:

$$\deg(\mathcal{N}_{\gamma'|X'_i}) = \deg(\mathcal{N}_{\gamma|X_i}) - 1, \quad 1 \leqslant i \leqslant 2.$$

Finally,

$$d_\gamma = d_{\gamma'}$$

and therefore we have

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) - 1 = d_\gamma \tag{10.20}$$

which proves the theorem in this case (i).

Consider now case (ii), i.e. p_1 is a double point for both E' and $\mathcal{X}'$.

If p_1 is an ordinary double point for $\mathcal{X}'$, we blow-up $\mathcal{X}'$ at p_1 and we get a new total space $\mathcal{X}''$. Let X''_1, X''_2 be the proper transforms of X'_1, X'_2, respectively, and let γ'' be the intersection of X''_1 and X''_2, which is isomorphic to γ. Notice that $\mathcal{X}''$ is smooth and the exceptional divisor E'' is a smooth quadric (see Figure 24).

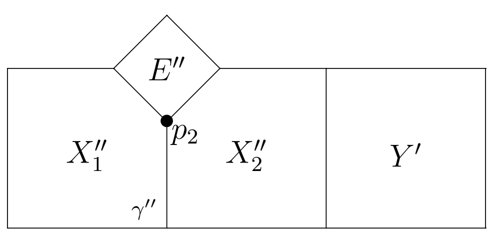


FIGURE 24. Blowing-up $\mathcal{X}'$ at p_1 when p_1 is ordinary for both $\mathcal{X}'$ and E'

We remark that the central fibre of $\mathcal{X}''$ is now non-reduced, since it contains E'' with multiplicity 2. Thus we apply Corollary 10.17 and we get

$$\mathcal{O}_{\gamma''} \cong \mathcal{N}_{\gamma''|X''_1} \otimes \mathcal{N}_{\gamma''|X''_2} \otimes \mathcal{O}_{\gamma''}(F_{\gamma''}).$$

Since,

$$\deg(\mathcal{N}_{\gamma''|X''_i}) = \deg(\mathcal{N}_{\gamma|X_i}) - 2, \ i = 1, 2, \qquad \deg F_{\gamma''} = f_3(\gamma) + 2,$$

then,

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) - 1 = d_\gamma + 1 > d_\gamma. \tag{10.21}$$

If the point p_1 is not an ordinary double point, we again blow-up p_1 as above. Now the exceptional divisor E'' of $\mathcal{X}''$ is a singular quadric in $\mathbb{P}^3$, which can only be either a quadric cone or it has to consist of two distinct planes E_1'', E_2''. Note that if p_1 lies on a double line of E' (i.e. p_1 is in the intersection of two irreducible components of E'), then only the latter case occurs since E'' has to contain a curve C_{E_4}.

Let $p_2 = E'' \cap \gamma''$. In the former case, if p_2 is not the vertex of the quadric cone, then the total space $\mathcal{X}''$ is smooth at p_2 and we can apply Corollary 10.17 and we get (10.21) as before.

If p_2 is the vertex of the quadric cone, then p_2 is a double point of $\mathcal{X}''$ and we can go on blowing-up $\mathcal{X}''$ at p_2. This blow-up procedure stops after finitely many, say h, steps and one sees that Formula (10.21) has to be replaced by

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) - 1 = d_\gamma + h > d_\gamma. \tag{10.22}$$

In the latter case, i.e. if E'' consists of two planes E_1'' and E_2'', let λ be the intersection line of E_1'' and E_2''. If p_2 does not belong to λ (see Figure 25), then p_2 is a smooth point of the total space $\mathcal{X}''$, therefore we can apply Corollary 10.17 and we get again Formula (10.21).

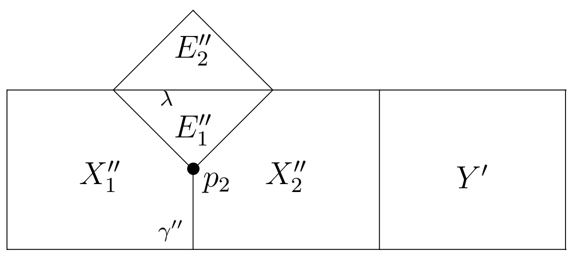


FIGURE 25. E'' splits in two planes E_1'', E_2'' and $p_2 \notin E_1'' \cap E_2''$

If p_2 lies on λ, then p_2 is a double point for the total space $\mathcal{X}''$ (see Figure 26). We can thus iterate the above procedure until the process terminates after finitely many, say h, steps by getting rid of the singularities which are infinitely near to p along γ. At the end, one gets again Formula (10.22). $\square$

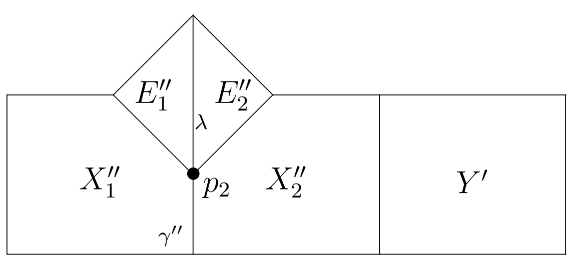


FIGURE 26. E''' splits in two planes E_1''', E_2''' and $p_3 \in E_1''' \cap E_2'''$

Remark 10.23. We observe that the proof of Theorem 10.2 proves a stronger result than what we stated in (10.3). Indeed, the idea of the proof is that we blow-up the total space $\mathcal{X}$ at each Zappatic singularity p in a sequence of singular points p, p_1, p_2, …, p_{h_p}, each infinitely near one to the other along γ. Note that p_i, $i = 1, \dots, h_p$, is a double point for the total space.

The above proof shows that the first inequality in (10.3) is an equality if and only if each Zappatic singularity of $\mathcal{X}$ has no infinitely near singular point. Moreover (10.22) implies that

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) - r_3(\gamma) - \sum_{n \geqslant 4} (\rho_n(\gamma) + f_n(\gamma)) = d_\gamma + \sum_{p \in \gamma} h_p.$$

In other words, as natural, every infinitely near double point along γ counts as a double point of the original total space along γ.

11. On some results of Zappa

In [55, 56, 57, 58, 59, 60, 61], Zappa considered degenerations of projective surfaces to a planar Zappatic surface with only R_3-, S_4- and E_3-points. One of the results of Zappa's analysis is that the invariants of a surface admitting a good planar Zappatic degeneration with mild singularities are severely restricted. In fact, translated in modern terms, his main result in [60] can be read as follows:

Theorem 11.1 (Zappa)**.** *Let $\mathcal{X} \to \Delta$ be a good, planar Zappatic degeneration, where the central fibre $\mathcal{X}_0 = X$ has at most R_3- and E_3-points. Then, for $t \neq 0$, one has*

$$K^2 := K^2_{\mathcal{X}_t} \leqslant 8\chi + 1 - g, \tag{11.2}$$

where $\chi = \chi(\mathcal{O}_{\mathcal{X}_t})$ and g is the sectional genus of $\mathcal{X}_t$.

Theorem 11.1 has the following interesting consequence:

Corollary 11.3 (Zappa)**.** *If $\mathcal{X} \to \Delta$ is a good, planar Zappatic degeneration of a scroll $\mathcal{X}_t$ of sectional genus $g \geqslant 2$ to $\mathcal{X}_0 = X$, then X has worse singularities than R_3- and E_3-points.*

Proof. For a scroll of genus g one has $8\chi + 1 - g - K^2 = 1 - g$. □

Actually Zappa conjectured that for most of the surfaces the inequality $K^2 \leqslant 8\chi + 1$ should hold and even proposed a plausibility argument for this. As well-known, the correct bound for all the surfaces is $K^2 \leqslant 9\chi$, proved by Miyaoka and Yau (see [40, 54]) several decades after Zappa.

We will see in a moment that Theorem 11.1 can be proved as consequence of the computation of K^2 (see Theorem 8.1) and the Multiple Point Formula (see Theorem 10.2).

Actually, Theorems 8.1 and 10.2 can be used to prove a stronger result than Theorem 11.1; indeed:

Theorem 11.4. *(cf.* [9, Theorem 8.4]*) Let $\mathcal{X} \to \Delta$ be a good, planar Zappatic degeneration, where the central fibre $\mathcal{X}_0 = X$ has at most R_3-, E_3-, E_4- and E_5-points. Then*

$$K^2 \leqslant 8\chi + 1 - g. \tag{11.5}$$

Moreover, the equality holds in (11.5) *if and only if $\mathcal{X}_t$ is either the Veronese surface in $\mathbb{P}^5$ degenerating to four planes with associated graph S_4 (i.e. with three R_3-points, see Figure 27.a), or an elliptic scroll of degree $n \geqslant 5$ in $\mathbb{P}^{n-1}$ degenerating to n planes with associated graph a cycle E_n (see Figure 27.b).*

Furthermore, if $\mathfrak{X}_t$ is a surface of general type, then

$$K^2 < 8\chi - g. \tag{11.6}$$

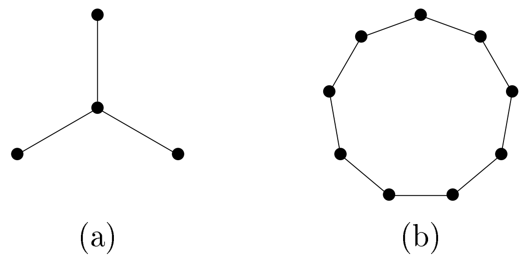


FIGURE 27.

Proof. Notice that if X has at most R_3-, E_3-, E_4- and E_5-points, then Formulas (8.3) and (8.5) give $K^2 = 9v - 10e + 6f_3 + 8f_4 + 10f_5 + r_3$. Thus, by (3.16) and (3.19), one gets

$$8\chi + 1 - g - K^2 = 8v - 8e + 8f_3 + 8f_4 + 8f_5 + 1 - (e - v + 1) - K^2 = e - r_3 + 2f_3 - 2f_5 =$$
$$= \frac{1}{2}(2e - 2r_3 + 3f_3 - 4f_4 - 5f_5) + \frac{1}{2}f_3 + 2f_4 + \frac{1}{2}f_5 \overset{(*)}{\geqslant} \frac{1}{2}f_3 + 2f_4 + \frac{1}{2}f_5 \geqslant 0$$

where the inequality $(*)$ follows from (10.6). This proves Formula (11.5) (and Theorem 11.1).

If $K^2 = 8\chi + 1 - g$, then $(*)$ is an equality, hence $f_3 = f_4 = f_5 = 0$ and $e = r_3$. Therefore, by Formula (3.23), we get

$$\sum_i w_i(w_i - 1) = 2r_3 = 2e, \tag{11.7}$$

where w_i denotes the valence of the vertex v_i in the graph G_X. By definition of valence, the right-hand-side of (11.7) equals $\sum_i w_i$. Therefore, we get

$$\sum_i w_i(w_i - 2) = 0. \tag{11.8}$$

If $w_i \geqslant 2$, for each $1 \leqslant i \leqslant v$, one easily shows that only the cycle as in Figure 27 (b) is possible. This gives

$$\chi = 0,\ K^2 = 0,\ g = 1,$$

which implies that $\mathfrak{X}_t$ is an elliptic scroll.

Easy combinatorial computations show that, if there is a vertex with valence $w_i \neq 2$, then there is exactly one vertex with valence 3 and three vertex of valence 1. Such a graph, with v vertices, is associated to a planar Zappatic surface of degree v in $\mathbb{P}^{v+1}$ with

$$\chi = 0,\ p_g = 0,\ g = 0.$$

Thus, by hypothesis, $K^2 = 9$ and, by properties of projective surfaces, the only possibility is that $v = 4$, G_X is as in Figure 27 (a) and $\mathfrak{X}_t$ is the Veronese surface in $\mathbb{P}^5$.

Suppose now that $\mathfrak{X}_t$ is of general type. Then $\chi \geqslant 1$ and $v = \deg(\mathfrak{X}_t) < 2g - 2$. Formulas (3.16) and (3.19) imply that $\chi = f - g + 1 \geqslant 1$, thus $f \geqslant g > v/2 + 1$. Clearly $v \geqslant 4$, hence $f \geqslant 3$. Proceeding as at the beginning of the proof, we have that:

$$8\chi - g - K^2 \geqslant \frac{1}{2}f_3 + 2f_4 + \frac{1}{2}f_5 - 1 \geqslant \frac{1}{2}f - 1 > 0,$$

or equivalently $K^2 < 8\chi - g$. □

Remark 11.9. By following the same argument of the proof of Theorem 11.4, one can list all the graphs and the corresponding smooth projective surfaces in the degeneration, for which $K^2 = 8\chi - g$. For example, one can find $\mathcal{X}_t$ as a rational normal scroll of degree n in $\mathbb{P}^{n+1}$ degenerating to n planes with associated graph a chain R_n. On the other hand, one can also have a del Pezzo surface of degree 7 in $\mathbb{P}^7$.

Let us state some applications of Theorem 11.4.

Corollary 11.10. *If $\mathcal{X}$ is a good, planar Zappatic degeneration of a scroll $\mathcal{X}_t$ of sectional genus $g \geqslant 2$ to $\mathcal{X}_0 = X$, then X has worse singularities than R_3-, E_3-, E_4- and E_5-points.*

Corollary 11.11. *If $\mathcal{X}$ is a good, planar Zappatic degeneration of a del Pezzo surface $\mathcal{X}_t$ of degree 8 in $\mathbb{P}^8$ to $\mathcal{X}_0 = X$, then X has worse singularities than R_3-, E_3-, E_4- and E_5-points.*

Proof. Just note that $K^2 = 8$ and $\chi = g = 1$, thus $\mathcal{X}_t$ satisfies the equality in (11.5). □

Corollary 11.12. *If $\mathcal{X}$ is a good, planar Zappatic degeneration of a minimal surface of general type $\mathcal{X}_t$ to $\mathcal{X}_0 = X$ with at most R_3-, E_3-, E_4- and E_5-points, then*

$$g \leqslant 6\chi + 5.$$

Proof. It directly follows from (11.6) and Noether's inequality, i.e. $K^2 \geqslant 2\chi - 6$. □

Corollary 11.13. *If $\mathcal{X}$ is a good planar Zappatic degeneration of an m-canonical surface of general type $\mathcal{X}_t$ to $\mathcal{X}_0 = X$ with at most R_3-, E_3-, E_4- and E_5-points, then*

(i) *$m \leqslant 6$;*
(ii) *if $m = 5, 6$, then $\chi = 3$, $K^2 = 1$;*
(iii) *if $m = 4$, then $\chi \leqslant 4$, $8\chi \geqslant 11K^2 + 2$;*
(iv) *if $m = 3$, then $\chi \leqslant 6$, $8\chi \geqslant 7K^2 + 2$;*
(v) *if $m = 2$, then $K^2 \leqslant 2\chi - 1$;*
(vi) *if $m = 1$, then $K^2 \leqslant 4\chi - 1$.*

Proof. Take $\mathcal{X}_t = S$ to be m-canonical. First of all, by Corollary 11.12, we immediately get (i). Then, by Formula (11.6), we get

$$8\chi - 2 \geqslant \frac{(m^2 + m + 2)}{2} K^2.$$

Thus, if m equals either 1 or 2, we find statements (v) and (vi).

Since S is of general type, by Noether's inequality we get

$$8\chi - 2 \geqslant (2\chi - 6)\frac{(m^2 + m + 2)}{2}.$$

This gives, for $m \geqslant 3$,

$$\chi \leqslant 3 + \frac{22}{(m^2 + m - 6)}$$

which, together with the above inequality, gives the other cases of the statement. □

It would be interesting to see whether the numerical cases listed in the above corollary can actually occur.

We remark that Corollary 11.10 implies in particular that one cannot hope to degenerate all surfaces to unions of planes with only global normal crossings, namely double lines and E_3-points; indeed, one needs at least E_n-points, for $n \geqslant 6$, or R_m-, S_m-points, for $m \geqslant 4$.

From this point of view, another important result of Zappa is the following (cf. [55, § 12]):

Theorem 11.14 (Zappa)**.** *For every $g \geqslant 2$ and every $d \geqslant 3g+2$, there are families of scrolls of sectional genus g, of degree d, with general moduli having a planar Zappatic degeneration with at most R_3-, S_4- and E_3-points.*

Zappa's arguments rely on a rather intricate analysis concerning degenerations of hyperplane sections of the scroll and, accordingly, of the branch curve of a general projection of the scroll to a plane. We have not been able to check all the details of this very clever argument. However, the idea which Zappa exploits, of degenerating the branch curve of a general projection to a plane, is a classical one which goes back to Enriques, Chisini, etc. and certainly deserves attention. In reading Zappa's paper [55], our attention has been attracted by other ingredients he uses to study the aforementioned degenerations, which look interesting on their own. Precisely, Zappa gives extendability conditions for a curve on a scroll which is not a cone. For an overview of these results with a modern approach, the reader is referred to [8, § 6]. In particular, we prove a slightly more general version of the following result of Zappa (cf. [8, Theorem 6.8]):

Proposition 11.15 (Zappa)**.** *Let $C \subset \mathbb{P}^2$ be a general element of the Severi variety $V_{d,g}$ of irreducible curves of degree d and geometric genus $g \geqslant 2$, with $d \geqslant 2g+2$. Then C is the plane section of a scroll $S \subset \mathbb{P}^3$ which is not a cone.*

Via completely different techniques, we prove in [8] the following result which generalizes Theorem 11.14 of Zappa:

Theorem 11.16. *(cf.* [8, Theorem 1.2]*) Let $g \geqslant 0$ and either $d \geqslant 2$, if $g = 0$, or $d \geqslant 5$, if $g = 1$, or $d \geqslant 2g+4$, if $g \geqslant 2$. Then there exists a unique irreducible component $\mathcal{H}_{d,g}$ of the Hilbert scheme of scrolls of degree d and sectional genus g in $\mathbb{P}^{d-2g+1}$, such that the general point of $\mathcal{H}_{d,g}$ represents a smooth scroll S which is linearly normal and moreover with $H^1(S, \mathcal{O}_S(1)) = 0$.*

Furthermore,

(i) *$\mathcal{H}_{d,g}$ is generically reduced and* $\dim(\mathcal{H}_{d,g}) = (d-2g+2)^2 + 7(g-1)$,

(ii) *$\mathcal{H}_{d,g}$ contains the Hilbert point of a planar Zappatic surface having only either $d-2$ R_3-points, if $g = 0$, or $d-2g+2$ points of type R_3 and $2g-2$ points of type S_4, if $g \geqslant 1$, as Zappatic singularities,*

(iii) *$\mathcal{H}_{d,g}$ dominates the moduli space $\mathcal{M}_g$ of smooth curves of genus g.*

For other results concerning the geometry of the general scroll parametrized by $\mathcal{H}_{d,g}$, cf. [11] and [12].

In a more general setting, it is a natural question to ask which Zappatic singularities are needed in order to degenerate as many smooth, projective surfaces as possible to good, planar Zappatic surface. Note that there are some examples (cf. §12) of smooth projective surfaces S which certainly cannot be degenerated to Zappatic surfaces with E_n-, R_n-, or S_n-points, unless n is large enough.

However, given such a S, the next result — i.e. Proposition 11.17 — suggests that there might be a birational model of S which can be Zappatically degenerated to a surface with only R_3- and E_n-points, for $n \leqslant 6$.

Proposition 11.17. *Let $\mathcal{X} \to \Delta$ be a good planar Zappatic degeneration and assume that the central fibre X has at most R_3- and E_m-points, for $m \leqslant 6$. Then*

$$K^2 \leqslant 9\chi.$$

Proof. The bounds for K^2 in Theorem 8.1 give $9\chi - K^2 = 9v - 9e + \sum_{m=3}^{6} 9f_m - K^2$. Therefore, we get:

$$2(9\chi - K^2) \geqslant 2e + 6f_3 + 2f_4 - 2f_5 - 6f_6 - 2r_3 \tag{11.18}$$

If we plug (10.6) in (11.18), we get

$$2(9\chi - K^2) \geqslant (2e + 3f_3 - 4f_4 - 5f_5 - 6f_6 - 2r_3) + (3f_3 + 6f_4 + 3f_5),$$

where both summands on the right-hand-side are non-negative. □

In other words, Proposition 11.17 states that the Miyaoka-Yau inequality holds for a smooth projective surface S which can degenerate to a good planar Zappatic surface with at most R_3- and E_n-points, $3 \leqslant n \leqslant 6$.

Another interesting application of the Multiple Point Formula is given by the following remark.

Remark 11.19. Let $\mathcal{X} \to \Delta$ be a good, planar Zappatic degeneration. Denote by δ the *class* of the general fibre $\mathcal{X}_t$ of $\mathcal{X}$, $t \neq 0$. By definition, δ is the degree of the dual variety of $\mathcal{X}_t$, $t \neq 0$. From Zeuthen-Segre (cf. [19] and [32]) and Noether's Formula (cf. [27], page 600), it follows that:

$$\delta = \chi_{\text{top}} + \deg(\mathcal{X}_t) + 4(g-1) = (9\chi - K^2) + 3f + e. \tag{11.20}$$

Therefore, (10.6) implies that:

$$\delta \geqslant 3f_3 + r_3 + \sum_{n \geqslant 4} (12-n) f_n + \sum_{n \geqslant 4} (n-1)\rho_n - k.$$

In particular, if X is assumed to have at most R_3- and E_3-points, then (11.20) becomes

$$\delta = (2e + 3f_3 - 2r_3) + (3f_3 + r_3),$$

where the first summand in the right-hand side is non-negative by the Multiple Point Formula; therefore, one gets

$$\delta \geqslant 3f_3 + r_3.$$

Zappa's original approach in [55], indeed, was to compute δ and then to deduce Formula (11.2) and Theorem 11.1 from this.

12. Examples of degenerations of surfaces to Zappatic unions of planes

The aim of this section is to illustrate some interesting examples of degenerations of smooth surfaces to good, planar Zappatic surfaces. We also discuss some examples of non-smoothable Zappatic surfaces and we pose open questions on the existence of degenerations to planar Zappatic surfaces for some classes of surfaces.

Product of curves (Zappa) (cf. Example 3.1 in [8]). Let $C \subset \mathbb{P}^{n-1}$ and $C' \subset \mathbb{P}^{m-1}$ be general enough curves. Consider the smooth surface

$$S = C \times C' \subset \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \subset \mathbb{P}^{nm-1}.$$

If C and C' can degenerate to stick curves, say to C_0 and C'_0 respectively, then the surface S degenerates to a union of quadrics Y with only double lines as singularities in codimension one and with Zappatic singularities.

If we could (independently) further degenerate each quadric of Y to the union of two planes, then we would be able to get a degeneration $\mathcal{X} \to \Delta$ with general fibre $\mathcal{X}_t \cong S = C \times C'$ and central fibre a good, planar Zappatic surface $\mathcal{X}_0 \cong X$. This certainly happens if each quadric of Y meets the other quadrics of Y along a union of lines of type (a, b), with $a, b \leqslant 2$ (see Figure 28).

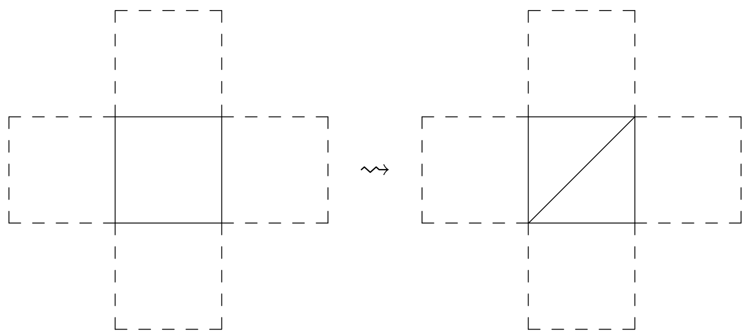


FIGURE 28. A quadric degenerating to the union of two planes

For example $S = C \times C'$ can be degenerated to a good, planar Zappatic surface if C and C' are either rational or elliptic normal curves and we degenerate them to stick curves C_{R_n} and C_{E_n}, respectively.

Let us see some of these cases in detail.

(a) *Rational normal scrolls* (cf. Example 3.2 in [8]). Let $C \subset \mathbb{P}^n$ be a rational normal curve of degree n. Since C can degenerate to a stick curve C_{R_n}, the surface $S = C \times \mathbb{P}^1 \subset \mathbb{P}^{2n+1}$ can degenerate to a chain of n quadrics as in Figure 29, which is a good Zappatic surface.

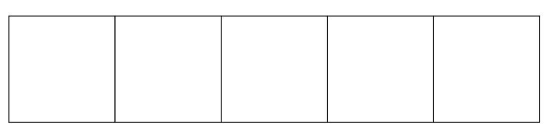


FIGURE 29. Union of n quadrics with a chain as associated graph

As we remarked above, the chain of quadrics can further degenerate to a good, planar Zappatic surface X which is the union of $2n$-planes (see Figure 30). Note that the surface X has only R_3-points as Zappatic singularities and its associated graph G_X is a chain R_{2n}. In this way, one gets degeneration of a rational normal scroll to a good, planar Zappatic surface with only R_3-points as Zappatic singularities.

(b) *Ruled surfaces* (cf. Example 3.3. in [8]). Let now $C \subset \mathbb{P}^{n-1}$ be a normal elliptic curve of degree n. Since C can degenerate to a stick curve C_{E_n}, the surface $S = C \times \mathbb{P}^1 \subset \mathbb{P}^{2n-1}$ can degenerate to a cycle of n quadrics (see Figure 31).

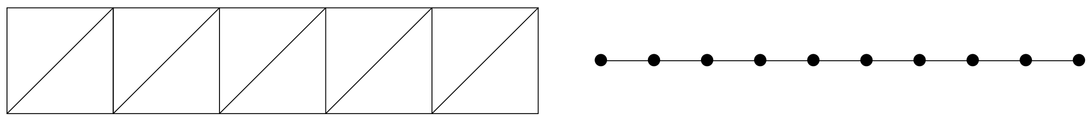


FIGURE 30. Union of $2n$-planes with a chain as associated graph

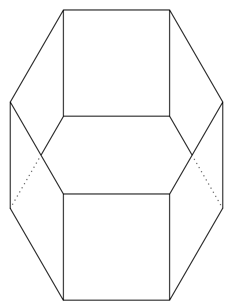


FIGURE 31. A cycle of n quadrics

As before, such a cycle of quadrics can degenerate to a good, planar Zappatic surface X which is a union of $2n$ planes with only R_3-points and whose associated graph corresponds to the cycle graph E_{2n+2} (see figure 32).

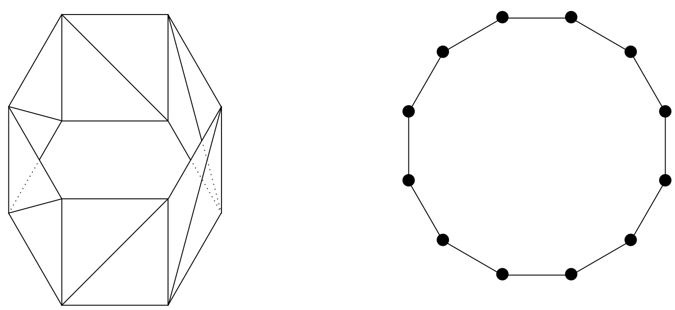


FIGURE 32. A union of $2n$ planes with a cycle as associated graph

(c) *Abelian surfaces* (cf. Example 3.4 in [8]). Let $C \subset \mathbb{P}^{n-1}$ and $C' \subset \mathbb{P}^{m-1}$ be smooth, elliptic normal curves of degree respectively n and m. Then C and C' degenerate to the stick curves C_{E_n} and C_{E_m} respectively, hence the abelian surface $S = C \times C' \subset \mathbb{P}^{nm-1}$ degenerates to a Zappatic surface which is a union of mn quadrics with only E_4-points as Zappatic singularities, cf. e.g. the picture on the left in Figure 33, where the top edges have to be identified with the bottom ones, similarly the left edges have to be identified with the right ones. Thus the top quadrics meet the bottom quadrics and the quadrics on the left meet the quadrics on the right.

Again each quadric degenerates to the union of two planes. By doing this as depicted in Figure 33, one gets a degeneration of a general abelian surface with a polarization of type (n, m) to a planar Zappatic surface of degree $2nm$ with only E_6-points as Zappatic singularities.

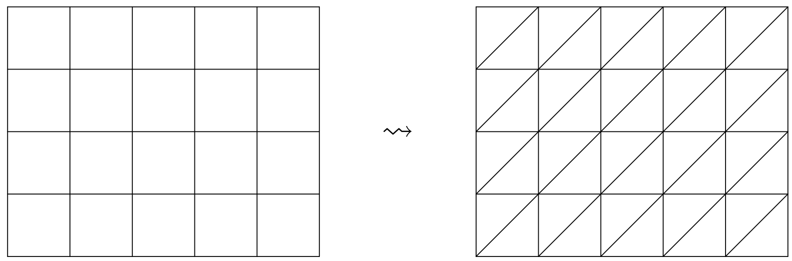


FIGURE 33. nm quadrics with E_4-points and $2nm$ planes with E_6-points

Concerning the general case, suppose that either C or C' has genus greater than 1. If C and C' degenerate to stick curves, then the surface $S = C \times C'$ degenerates as above to a union of quadrics. Unfortunately it is not clear if it is possible to further independently degenerate each quadric to two planes in the same way as above and, in [58], Zappa left as an open problem to prove the degeneration to a union of planes.

Theorem 11.16, proved in the paper [8], show that $C \times \mathbb{P}^1$, suitably embedded as a non-special, linearly normal scroll, really degenerates to a union of planes with only R_3- and S_4-points. Indeed, let C be any curve of genus g and let L be a very-ample non-special line bundle of degree $d \geqslant g + 3$. The global sections of L determine an embedding of C in $\mathbb{P}^{d-g}$. Consider the Segre embedding of $C \times \mathbb{P}^1$. This gives a non-special, linearly normal, smooth scroll S of degree $2d$ in $\mathbb{P}^{2d-2g+1}$ and the corresponding point sits in the irreducible component $\mathcal{H}_{2d,g}$ of the Hilbert scheme mentioned in Theorem 11.16. By our results in [8], it follows that S degenerates to a planar Zappatic surface with $2(d - g + 1)$ points of type R_3 and $2(g - 1)$ points of type S_4 (cf. Construction 4.2 in [8]).

Degeneration to cones. Recall the following result of Pinkham:

Theorem 12.1 (Pinkham [47, 48])**.** *Let $S \subset \mathbb{P}^n$ be a smooth, irreducible and projectively Cohen-Macaulay surface. Then S degenerates to the cone over a hyperplane section of S.*

Let C be the hyperplane section of S. Suppose that C can be degenerated to a stick curve C_0. In this case, S can be degenerated to the cone X over the stick curve C_0. By definition, X is a Zappatic surface only if C has genus either 0 or 1. Therefore:

Corollary 12.2. *(i) Any surface S of minimal degree (i.e. of degree n) in $\mathbb{P}^{n+1}$ can be degenerated to the cone over the stick curve C_{T_n}, for any tree T_n with n vertices (cf. Example 2.7).*
(ii) Any del Pezzo surface S of degree n in $\mathbb{P}^n$, $n \leqslant 9$, can be degenerated to the cone over the stick curve C_{Z_n}, for any connected graph with $n \geqslant 3$ vertices and $h^1(Z_n, \mathbb{C}) = 1$ (cf. Example 2.8).

For $n = 4$, recall that the surfaces of minimal degree in $\mathbb{P}^5$ are either the Veronese surface (which has $K^2 = 9$) or a rational normal scroll (which has $K^2 = 8$). Therefore:

Corollary 12.3 (Pinkham)**.** *The local deformation space of a T_4-singularity is reducible.*

Veronese surfaces (Moishezon-Teicher). Consider $V_d \subset \mathbb{P}^{d(d+3)/2}$ be the d-Veronese surface, namely the embedding of $\mathbb{P}^2$ via the linear system $|\mathcal{O}_{\mathbb{P}^2}(d)|$. In [42], Moishezon and

Teicher described a "triangular" degeneration of V_d such that the central fibre is a union of d^2 planes with only R_3- and E_6-points as Zappatic singularities (see Figure 34).

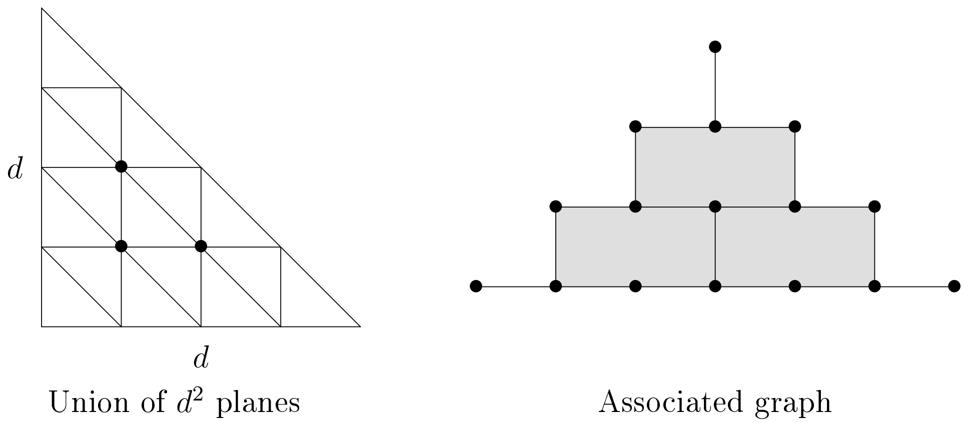


FIGURE 34. Degeneration of the d-Veronese surface.

Let us explain how to get such a degeneration. Consider the trivial family $\mathcal{X} = \mathbb{P}^2 \times \Delta$, where Δ is a complex disk. Let $\mathcal{L} \cong \mathcal{O}_{\mathcal{X}}(d)$ be a line bundle on $\mathcal{X}$. If we blow-up a point in the central fibre of $\mathcal{X}$, the new central fibre becomes as the left-hand-side picture in Figure 35, where E is the exceptional divisor, $\Pi = \mathbb{P}^2$ and $\mathbb{F}_1$ is the Hirzebruch surface.

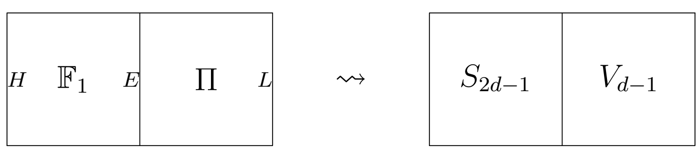


FIGURE 35. Degenerating the d-Veronese surface V_d

Let $\tilde{\mathcal{X}}$ be the blown-up family and let $\tilde{\mathcal{L}}$ be the line bundle on $\tilde{\mathcal{X}}$ given by the pull-back of $\mathcal{L}$ twisted by the divisor $-(d-1)\Pi$. Then $\tilde{\mathcal{L}}$ restricts to $dH-(d-1)E$ on the surface $\mathbb{F}_1$ and to $(d-1)L$ on the plane Π, respectively.

These line bundles embed, respectively, $\mathbb{F}_1$ as a rational normal scroll S_{2d-1} of degree $2d-1$ and Π as the $(d-1)$-Veronese surface V_{d-1} meeting along a rational normal curve of degree $d-1$ (see the right-hand-side picture of Figure 35). One can independently degenerate V_{d-1} ro a good, planar Zappatic surface, by induction on d (getting the left most bottom triangle in figure 34) and the rational normal scroll S_{2d-1} as we saw before (getting the top strip in the triangle in figure 34).

K3 surfaces. In the paper [15], the authors construct a specific projective degeneration of a $K3$ surface of degree $2g-2$ in $\mathbb{P}^g$ to a planar Zappatic surface which is a union of $2g-2$ planes, which meet in such a way that the associated graph to the configuration of planes is a triangulation of the 2-sphere.

In the previous paper [14], planar Zappatic degenerations of $K3$ surfaces were constructed in such a way that the general member of the degeneration was embedded by a primitive

line bundle. In [15] the general member of the degeneration is embedded by a multiple of a primitive line bundle class (for details, the reader is referred to the original articles).

Let X denote a good, planar Zappatic surface which is a degeneration of a $K3$-surface $S \subset \mathbb{P}^g$ of genus g and let G be the associated graph to X. Then, G is planar, because $p_g(S) = 1$, and 3-valent (see [14]). By using Notation 3.12, we get

$$v = 2g - 2, \qquad e = 3g - 3, \qquad f = g + 1. \tag{12.4}$$

Conversely, by starting from a planar graph G with invariants as in (12.4), one can find a Zappatic numerical $K3$ surface X whose associated graph is G. Such an X is called a *graph surface*. Smoothable graph surfaces are exhibited in [14] and [15].

The specific degenerations constructed in [15] depend on two parameters and can be viewed as two rectangular arrays of planes, joined along their boundary. For this reason, these are called *pillow degenerations*.

Take two integers a and b greater than or equal to two and set $g = 2ab + 1$. The number of planes in the pillow degeneration is then $2g - 2 = 4ab$. The projective space $\mathbb{P}^g$ has $g + 1 = 2ab + 2$ coordinate points, and each of the $4ab$ planes is obtained as the span of three of these. These sets of three points are indicated in Figure 36, which describes the bottom part of the "pillow" and the top part of the "pillow", which are identified along the boundaries of the two configurations. The reader will see that the boundary is a cycle of $2a + 2b$ lines.

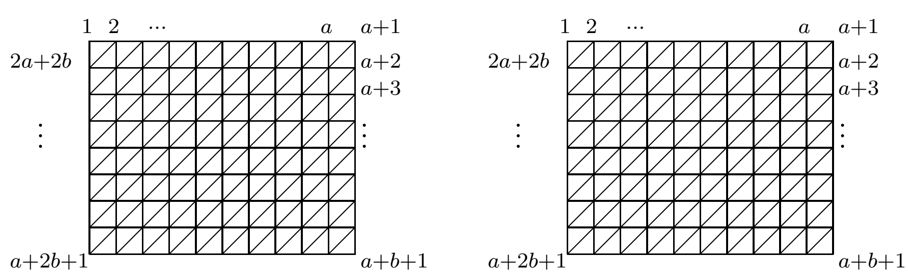
FIGURE 36. Configuration of Planes, Top and Bottom

Top

Boundary points labeled from 1 through $2a + 2b$, clockwise; interior points labeled from $2a + 2b + 1$ through $ab + a + b + 1$

Bottom

Boundary points labeled from 1 through $2a + 2b$, clockwise; interior points labeled from $ab + a + b + 2$ through $2ab + 2$



Note that no three of the planes meet along a line. Also note that the set of bottom planes lies in a projective space of dimension $ab + a + b$, as does the set of top planes; these two projective spaces meet exactly along the span of the $2a + 2b$ boundary points, which has dimension $2a + 2b - 1$. Finally note that the four corner points of the pillow degeneration (labelled 1, $a{+}1$, $a{+}b{+}1$, and $a{+}2b{+}1$) are each contained in three distinct planes, whereas all the other points are each contained in six planes. This property, that the number of lines and planes incident on each of the points is bounded is a feature of the pillow degeneration that is not available in other previous degenerations (see [14]). We will call such a configuration of planes a *pillow of bidegree* (a, b).

Observe that a pillow of bidegree (a, b) is a planar Zappatic surface of degree $2g - 2$, having four R_3-points and $2ab - 2 = g - 3$ E_6-points as Zappatic singularities.

Remark 12.5. E_n-points, with $n \geqslant 6$, are unavoidable for the degeneration of $K3$ surfaces with hyperplane sections of genus g, if $g \geqslant 12$. Indeed, by using Notation 3.12, Formula (12.4) and the fact that G is 3-valent, we get

$$g+1 = f = \sum_n f_n \quad \text{and} \quad 6(g-1) = 3v = \sum_n n f_n.$$

These give

$$\sum_n (6-n) f_n = 12.$$

If we assume that $f_n = 0$, for $n \geqslant 6$, the last equality gives $2f_4 + f_5 = 12$ and so $10f_4 + 5f_5 = 60$. This equality, together with $4f_4 + 5f_5 = 6g - 6$ gives $g + f_4 = 11$, i.e. $g \leqslant 11$.

Complete intersections. Consider a surface $S \subset \mathbb{P}^n$ which is a *general* complete intersection of type $(d_1, \ldots, d_{n-2})$. Namely S is defined as the zero-locus

$$f_1 = \cdots = f_{n-2} = 0,$$

where f_i is a general homogeneous polynomial of degree d_i, $1 \leqslant i \leqslant n-2$.

One can degenerate any hypersurface $f_i = 0$ to the union of d_i hyperplanes. This implies that S degenerates to a planar Zappatic surface X with global normal crossings, i.e. with only E_3-points as Zappatic singularities.

We remark that degenerations of surfaces to good, planar Zappatic ones are possible also when S is projectively Cohen-Macaulay in $\mathbb{P}^4$, by some results of Gaeta (see [23]).

Non-smoothable Zappatic surfaces. The results of the previous sections allow us to exhibit simple planar Zappatic surfaces which are not smoothable, i.e. which cannot be the central fibre of a degeneration. For example, the planar Zappatic surface X with the graph of Figure 37 as associated graph is not smoothable. Indeed, if X were the central fibre of a degeneration $\mathcal{X} \to \Delta$, then Formulas (8.2) and (8.3) would imply

$$9 \leqslant K^2 \leqslant 10,$$

which is absurd because of the classification of smooth projective surface of degree 5 in $\mathbb{P}^6$ (see Theorem 6.18).

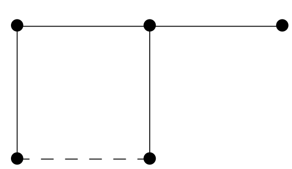


FIGURE 37. A non-smoothable planar Zappatic surface

It is an interesting problem to find more examples of smoothable Zappatic surfaces with only R_3- and E_n-points, $3 \leqslant n \leqslant 6$.

E.g., does there exist a Zappatic degeneration with only R_3 and E_n, $3 \leqslant n \leqslant 6$, for Enriques' surfaces?

Appendix A. Normal, Cohen-Macaulay and Gorenstein properties

The aim of this appendix is to briefly recall some well-known terminology and results concerning normal, Cohen-Macaulay and Gorenstein properties for projective varieties. These results are frequently used in Section 6. In the sequel, the term variety does not imply that the scheme under consideration is supposed to be irreducible.

First, we focus on "normality" conditions.

Definition A.1. An algebraic scheme X is said to be *normal* if, for each $p \in X$, the local ring $\mathcal{O}_{X,p}$ is an integrally closed domain (cf. [30], pp. 23 and 91).

Recall that if A is a local ring, with maximal ideal m, the *depth* of A is the maximal length of a regular sequence $x_1, \ldots, x_r$ with all $x_i \in m$ (cf. [30], page 184).

Recall that, by the theorem of Krull-Serre (see, e.g., [30], Theorem 8.22A, page 185), X is normal at p if, and only if, X is non-singular in codimension one at p and the local ring $\mathcal{O}_{X,p}$ has the S_2-property, i.e. its depth is greater than or equal to 2.

Let $X \subset \mathbb{P}^r$ be a projective variety and let

$$I(X) := H^0_*(\mathcal{I}_{X|\mathbb{P}^r}) = \bigoplus_{n\in\mathbb{Z}} H^0(\mathcal{I}_{X|\mathbb{P}^r}(n)) \tag{A.2}$$

be the saturated ideal associated to X in the homogeneous polynomial ring $S := \mathbb{C}[x_0, \ldots, x_r]$.

Definition A.3. A projective variety $X \subset \mathbb{P}^r$ is said to be *projectively normal* (with respect to the given embedding) if its homogeneous coordinate ring

$$\Gamma(X) := \mathbb{C}[x_0, \ldots, x_r]/I(X) \tag{A.4}$$

is an integrally closed domain (cf. [30], pp. 23 and 126).

Recall that projective normality is a property of the given embedding $X \subset \mathbb{P}^r$ and not only of X. Observe also that if X is a projectively normal variety, then it is also irreducible and normal (see [30], page 23).

If $X \subset \mathbb{P}^r$ is a projective variety, we denote by

$$\mathcal{C}(X) \subset \mathbb{A}^{r+1} \tag{A.5}$$

the affine cone over X having vertex at the origin $\underline{o} \in \mathbb{A}^{r+1}$. Observe that the homogeneous coordinate ring $\Gamma(X)$ in (A.4) coincides with the coordinate ring of the affine cone $\mathcal{C}(X)$.

We can characterize projective normality of a closed projective variety $X \subset \mathbb{P}^r$ in terms of its affine cone $\mathcal{C}(X)$.

Proposition A.6. *Let $X \subset \mathbb{P}^r$ be a projective variety and let $\mathcal{C}(X)$ be its affine cone. Then, the following conditions are equivalent:*

(i) *X is projectively normal;*
(ii) *$\mathcal{C}(X)$ is normal;*
(iii) *$\mathcal{C}(X)$ is normal at the vertex $\underline{o}$.*

Proof. See [26], Prop. 7.10, p. 57, and [30], p. 147. □

Remark A.7. The above proposition is an instance of a general philosophy which states that "the properties of the vertex of $\mathcal{C}(X)$ are equivalent to global properties of $\mathcal{C}(X)$ as well as of X" (cf. [26], page 54).

Proposition A.8. *Let $X \subset \mathbb{P}^r$ be a projective variety and let H be a hyperplane section of X. If H is projectively normal, then X is projectively normal.*

Proof. It is a trivial consequence of Krull-Serre's theorem (see, e.g., [26], Theorem 4.27). □

Definition A.9. A projective variety $X \subset \mathbb{P}^r$ is called *arithmetically normal* if the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^r}(j)) \to H^0(\mathcal{O}_X(j)) \tag{A.10}$$

is surjective, for every $j \in \mathbb{N}$.

Remark A.11. The surjectivity of the map in (A.10) is equivalent to

$$H^1(\mathcal{I}_{X|\mathbb{P}^r}(j)) = 0, \qquad \text{for every } j \in \mathbb{Z}. \tag{A.12}$$

This follows from the sequence:

$$0 \to \mathcal{I}_{X|\mathbb{P}^r}(j) \to \mathcal{O}_{\mathbb{P}^r}(j) \to \mathcal{O}_X(j) \to 0 \tag{A.13}$$

and from the cohomology of projective spaces.

One has the following relationship among the above three notions:

Proposition A.14. *$X \subset \mathbb{P}^r$ is projectively normal if, and only if, X is normal and arithmetically normal in $\mathbb{P}^r$ (cf. [30], pg. 126, and [62]).*

A fundamental property related to arithmetical normality is the following:

Proposition A.15. *If a hyperplane section H of a projective variety X is arithmetically normal, then X is arithmetically normal.*

Proof. Consider the following commutative diagram with two short exact sequences of sheaves:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(j-1) & \longrightarrow & \mathcal{O}_X(j) & \longrightarrow & \mathcal{O}_H(j) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^r}(j-1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^r}(j) & \longrightarrow & \mathcal{O}_{\mathbb{P}^{r-1}}(j) & \longrightarrow & 0 \end{array} \tag{A.16}$$

for all $j \in \mathbb{Z}$, where the vertical arrows are defined by the usual short exact sequence (A.13). Diagram (A.16) induces in cohomology the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{O}_X(j)) & \xrightarrow{\gamma_j} & H^0(\mathcal{O}_H(j)) \\ {\scriptstyle \delta_j}\uparrow & & \uparrow\!\!\uparrow{\scriptstyle \beta_j} \\ H^0(\mathcal{O}_{\mathbb{P}^r}(j)) & \xrightarrow{\alpha_j} & H^0(\mathcal{O}_{\mathbb{P}^{r-1}}(j)) \end{array}$$

for every $j \in \mathbb{Z}$, where α_j is trivially surjective and β_j is surjective by hypothesis. The surjectivity of the composite map $\gamma_j \circ \delta_j = \beta_j \circ \alpha_j$ forces the surjectivity of γ_j, but not yet that of δ_j. However, we have the injection:

$$H^1(\mathcal{O}_X(j-1)) \hookrightarrow H^1(\mathcal{O}_X(j)),$$

for every $j \in \mathbb{Z}$, which implies

$$H^1(\mathcal{O}_X(j)) = 0, \qquad \text{for each } j, \tag{A.17}$$

because $H^1(\mathcal{O}_X(j)) = 0$ for $j \gg 0$ by Serre's Theorem. The long exact sequences in cohomology induced by (A.16) then become:

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{O}_X(j-1)) \arrow[r] & H^0(\mathcal{O}_X(j)) \arrow[r] & H^0(\mathcal{O}_H(j)) \arrow[r] & 0 \\
0 \arrow[r] & H^0(\mathcal{O}_{\mathbb{P}^r}(j-1)) \arrow[r] \arrow[u, "\delta_{j-1}"] & H^0(\mathcal{O}_{\mathbb{P}^r}(j)) \arrow[r] \arrow[u, "\delta_j"] & H^0(\mathcal{O}_{\mathbb{P}^{r-1}}(j)) \arrow[r] \arrow[u, two heads, "\beta_j"] & 0
\end{tikzcd}$$

where we recall that β_j is surjective for all j by hypothesis. The map δ_j is trivially surjective for $j \leqslant 0$. Since $\delta_{j-1} = \delta_0$ is surjective for $j = 1$, the map $\delta_j = \delta_1$ is surjective too by elementary diagram chase. Hence we conclude by induction on j. □

We consider now "Cohen-Macaulay" conditions.

Recall that a local ring (A, m) is said to be *Cohen-Macaulay* (*CM* for short) if $\operatorname{depth}(A) = \dim(A)$ (see [30], page 184). This is equivalent to saying that a zero-dimensional local ring is always CM and, if $\dim(A) > 0$, then A is CM if, and only if, there is a non zero-divisor x in A such that $A/(x)$ is CM. In this case, $A/(x)$ is CM for every non-zero divisor x in A (cf. [18], [37] page 107).

Remark A.18. Let (R, M) be a regular local ring. Let (A, m) be a local R-algebra which is finitely generated as a R-module. Set $c := \operatorname{codim}_R(A)$. One has that (A, m) is CM if, and only if, there is a minimal, free resolution of R-modules

$$\mathcal{F}\colon 0 \to F_c \to F_{c-1} \to \cdots \to F_1 \to R \to A \to 0, \tag{A.19}$$

(cf. Corollary 21.16 and the Auslander-Buchsbaum Formula Theorem 19.9 in [18]).

If (A, m) is CM, then one defines

$$\omega_A := Ext^c_R(A, R) \tag{A.20}$$

to be the *canonical module* of A (cf. Theorem 21.15 in [18]). Then, the resolution (A.19) is such that $\mathcal{F}^*$ is a minimal free resolution of ω_A (cf. Corollary 21.16 [18]).

A finitely generated $\mathbb{C}$-algebra B is said to be *Cohen-Macaulay* (*CM* for short) if, and only if, for every prime ideal p of B, B_p is a CM local ring. This is equivalent to saying that B_p is a CM local ring, for every maximal ideal p in B (cf. [18], Prop. 18.8).

Definition A.21. (cf. [30], page 185) An algebraic, equidimensional scheme X is *Cohen-Macaulay at a point* $p \in X$ (*CM at* p, for short) if $\mathcal{O}_{X,p}$ is a Cohen-Macaulay, local ring. X is *Cohen-Macaulay* (*CM*, for short) if X is Cohen-Macaulay at each $p \in X$.

We have the following result:

Theorem A.22. *Let $X \subset \mathbb{P}^r$ be an equidimensional projective variety and let p be a point of X. Then, the following conditions are equivalent:*

(i) *X is CM at p;*
(ii) *each equidimensional hyperplane section of X through p is CM at p;*

Proof. It directly follows from the definition of local, CM rings. □

Definition A.23. A projective, equidimensional variety $X \subset \mathbb{P}^r$ is said to be *projectively Cohen-Macaulay* (*pCM*, for short) if the ring $\Gamma(X)$ as in (A.4) is a Cohen-Macaulay ring.

Definition A.24. A projective and equidimensional variety $X \subset \mathbb{P}^r$ is said to be *arithmetically Cohen-Macaulay* (*aCM*, for short), if X is arithmetically normal and moreover

$$H^i(\mathcal{O}_X(j)) = 0, \qquad \text{for every } j \in \mathbb{Z} \text{ and } 1 \leqslant i \leqslant n-1, \tag{A.25}$$

where $n = \dim(X)$.

Remark A.26. By standard exact sequences, X is arithmetically Cohen-Macaulay iff

$$H^i(\mathcal{I}_{X|\mathbb{P}^r}(j)) = 0, \qquad \text{for every } j \in \mathbb{Z} \text{ and } 1 \leqslant i \leqslant n. \tag{A.27}$$

Remark A.28. For $n = 1$, Formula (A.25) trivially holds. Thus a curve is arithmetically Cohen-Macaulay if and only if it is arithmetically normal.

As in Proposition A.6, we have:

Proposition A.29. *Let $X \subset \mathbb{P}^r$ be an equidimensional variety, $\mathcal{C}(X) \subset \mathbb{A}^{r+1}$ be the affine cone over X and let $\underline{o}$ be its vertex. Then, the following conditions are equivalent:*

(i) *X is pCM;*
(ii) *$\mathcal{C}(X)$ is CM;*
(iii) *$\mathcal{C}(X)$ is CM at $\underline{o}$ (cf. Remark A.7).*

Proof. Take $\Gamma(X)$ as in (A.4). Then, it coincides with the coordinate ring of the affine cone $\mathcal{C}(X) \subset \mathbb{A}^{r+1}$. Thus, the claim follows from Proposition 18.8 and Ex. 19.10 in [18]. □

Proposition A.30. *Let $X \subset \mathbb{P}^r$ be an equidimensional variety. Then:*

(i) *X is pCM $\Leftrightarrow$ X is aCM;*
(ii) *X is aCM $\Rightarrow$ X is CM.*

Proof. (i) From Proposition A.29, X is pCM iff $\mathcal{C}(X)$ is CM at $\underline{o}$. From [30], page 217, Ex. 3.4 (b), this implies that $H^i_{\underline{m}}(\Gamma(X)) = 0$, for all $i < r - c = n$, where $c = \operatorname{codim}_{\mathbb{P}^r}(X)$, $n = \dim(X)$, $\underline{m}$ is the maximal, homogeneous ideal of $\Gamma(X)$ and, as usual, $H^i_{\underline{m}}(-)$ is the local cohomology (see [30], Ex. 3.3 (a), page 217).

On the other hand, from [18], Theorem A4.1,

$$H^i_{\underline{m}}(\Gamma(X)) \cong \bigoplus_{j \in \mathbb{Z}} H^i(X, \mathcal{O}_X(j)).$$

(ii) Since X is covered by affine open subsets which are hyperplane sections of $\mathcal{C}(X)$, the assertion follows from Theorem A.22.

□

Theorem A.31. *Let $X \subset \mathbb{P}^r$ be an equidimensional closed subscheme in $\mathbb{P}^r$. Then X is aCM if, and only if, any hyperplane section H of X not containing any component of X is aCM.*

Proof. It directly follows from Theorem A.22 and Proposition A.29. □

Proposition A.32. *If a curve H, which is a hyperplane section of a projective surface S, is arithmetically normal then S is aCM (equiv., pCM).*

Proof. To prove that S is aCM, one has to prove that (A.17) holds (see Remark A.28). This follows by the proof of Proposition A.15. □

Corollary A.33. *Let $X \subset \mathbb{P}^r$ be a curve. Then:*
(i) X is projectively normal $\Rightarrow$ X is pCM.
(ii) If, furthermore, X is assumed to be smooth, the implication in (i) is an equivalence.

Proof. By Proposition A.30, X is pCM iff is aCM. On the other hand, since X is a curve, by Remark A.28 X aCM is equivalent to X arithmetically normal. Therefore, the curve X is in particular pCM if, and only if, it is arithmetically normal. Only if X is also smooth, then X is projectively normal, as it follows from Proposition A.14. □

Proposition A.34. *Let $X \subset \mathbb{P}^r$ be a projective, equidimensional variety s.t.* $\mathrm{codim}_{\mathbb{P}^r}(X) = c$. *Then, the following conditions are equivalent:*

(i) *X is pCM;*
(ii) *the* projective dimension *of $\Gamma(X)$ is equal to $c = \mathrm{codim}_{\mathbb{P}^r}(X)$. In other words, there is a minimal graded free resolution of $\Gamma(X)$,*

$$0 \to F_c \to F_{c-1} \to \cdots \to F_1 \to S \to \Gamma(X) \to 0 \tag{A.35}$$

where F_i is a free S-module, for $1 \leqslant i \leqslant c$.

Proof. From Proposition A.29, X is pCM if, and only if, $\Gamma(X)$ is CM. Let $S = \mathbb{C}[x_0, \ldots, x_r]$ be the homogeneous graded polynomial ring which is a finitely generated algebra over $\mathbb{C}$. Let M be the homogeneous maximal ideal in S. Since $\Gamma(X)$ is a finitely generated graded S module of finite projective dimension, then by the *Auslander-Buchsbaum Formula* in the graded case, we have

$$\mathrm{pd}_S(\Gamma(X)) = \mathrm{depth}_M(S) - \mathrm{depth}_{M\Gamma(X)}(\Gamma(X)) = r + 1 - \mathrm{depth}_{M\Gamma(X)}(\Gamma(X)), \tag{A.36}$$

where $\mathrm{pd}_S(\Gamma(X))$ is the projective dimension of $\Gamma(X)$ (cf. [18], Ex. 19.8, page 485).

Thus, if X is pCM, then $\Gamma(X)$ is CM and therefore $\mathrm{depth}_{M\Gamma(X)}(\Gamma(X)) = n+1$ so, by (A.36), $\mathrm{pd}_S(\Gamma(X)) = c = \mathrm{codim}_{\mathbb{P}^r}(X)$. Conversely, if $\mathrm{pd}_S(\Gamma(X)) = c$, then $\mathrm{depth}_{M\Gamma(X)}(\Gamma(X)) = n+1$, hence $\mathcal{C}(X)$ is CM at $\underline{o}$. One concludes by Proposition A.29. □

Remark A.37. Observe that the ranks of the free modules F_i, $1 \leqslant i \leqslant c$, do not depend on the minimal free resolution of $\Gamma(X)$. In particular, the rank of F_c in any minimal free resolution of $\Gamma(X)$ is an invariant of $\Gamma(X)$ called the *Cohen-Macaulay type* of X.

We now focus on "Gorenstein" conditions. First, we recall some standard definitions.

Definition A.38. Let (A, m) be a local, CM ring with residue field K.

If $\dim(A) = 0$, then A is called a *Gorenstein* ring if, and only if,

$$A \cong Hom_K(A, K). \tag{A.39}$$

If $\dim(A) > 0$, then A is called a *Gorenstein* ring if, and only if, there is a non-zero divisor $x \in A$ s.t. $A/(x)$ is Gorenstein. In this case, for every non-zero divisor $x \in A$, $A/(x)$ is Gorenstein.

Remark A.40. By using (A.20), (A, m) is Gorenstein if, and only if, $\omega_A \cong A$ (cf. [18], Theorem 21.15). As in in Remark A.18, this is equivalent to saying that there is a minimal, free resolution of R-modules

$$\mathcal{F} \; : \; 0 \to F_c \to F_{c-1} \to \ldots F_1 \to R \to A \to 0, \tag{A.41}$$

which is symmetric in the sense that $\mathcal{F} \cong \mathcal{F}^*$. This, in turn, is equivalent to saying that $F_c \cong R$ (see Corollary 21.16 from [18]).

Observe that, if (A, m) is Gorenstein then, for each prime ideal p in A, (A_p, pA_p) is Gorenstein. This follows by the above remark and by the flatness of localization (see [18], page 66).

Definition A.42. Let K be a field. Let R be a graded, finitely generated K-algebra.

If $\dim(R) = 0$, then R is called a *Gorenstein* graded ring if, and only if, there is an integer δ such that

$$R(\delta) \cong Hom_K(R, K). \tag{A.43}$$

If $\dim(R) > 0$, then R is called a *Gorenstein* graded ring if, and only if, there is an homogeneous non-zero divisor $x \in R$ such that $R/(x)$ is Gorenstein.

Remark A.44. Let $S = \mathbb{C}[x_0, \dots, x_{r+1}]$ be the homogeneous polynomial ring and let I be a homogeneous ideal, such that $A = S/I$ is a CM ring. Set $c = \mathrm{codim}_S(A)$.

Then

$$\omega_A := Ext^c_S(A, S(-r-1)) \tag{A.45}$$

is called the graded *dual module* of A. Then, A is Gorenstein if, and only if, there is an integer δ such that $\omega_A \cong A(\delta)$. This is equivalent to saying that there is a minimal, free graded resolution

$$\mathcal{F}\colon 0 \to F_c \to F_{c-1} \to \cdots \to F_1 \to S \to A \to 0, \tag{A.46}$$

which is symmetric in the sense that $\mathcal{F} \cong \mathcal{F}^*$. This, in turn, is equivalent to saying that there is an integer γ such that $F_c \cong S(\gamma)$ (see the proof of Corollary 21.16 and §21.11 from [18]).

Observe also that, if A as above is Gorenstein then, for every prime ideal P in A, (A_P, PA_P) is a local Gorenstein ring.

In complete analogy with Definition A.21, we have:

Definition A.47. A projective scheme X is *Gorenstein* at a point $p \in X$ if $\mathcal{O}_{X,p}$ is a Gorenstein, local ring. X is *Gorenstein*, if it is Gorenstein at each point $p \in X$.

Theorem A.48. *Let $X \subset \mathbb{P}^r$ be an equidimensional projective variety and let p be a point of X. Then, the following conditions are equivalent:*

(i) *X is Gorenstein at p;*
(ii) *each equidimensional hyperplane section of X through p is Gorenstein.*

Proof. It directly follows from the definition of local, Gorenstein rings. □

Definition A.49. A projective and equidimensional variety $X \subset \mathbb{P}^r$ is called *projectively Gorenstein* (*pG*, for short) if its homogeneous coordinate ring $\Gamma(X)$ is Gorenstein.

In complete analogy with Proposition A.29, we have:

Proposition A.50. *Let $X \subset \mathbb{P}^r$ be an equidimensional variety, let $\mathcal{C}(X) \subset \mathbb{A}^{r+1}$ be the affine cone over X and let $\underline{o}$ be its vertex. Then, the following conditions are equivalent:*

(i) *X is pG;*
(ii) *$\mathcal{C}(X)$ is Gorenstein;*
(iii) *$\mathcal{C}(X)$ is Gorenstein at $\underline{o}$;*
(iv) *X is pCM and the dualizing sheaf*

$$\omega_X \cong \mathcal{O}_X(a), \qquad \textit{for some } a \in \mathbb{Z}. \tag{A.51}$$

Proof. We prove the following implications.

- (i) $\Rightarrow$ (ii): it directly follows from what recalled in Remark A.44;
- (ii) $\Rightarrow$ (iii): trivial;
- (iii) $\Rightarrow$ (i): X is pCM from Proposition A.29. Now, let

$$\mathcal{F} \ : \ 0 \to F_c \to F_{c-1} \to \ldots F_1 \to S \to \Gamma(X) \to 0$$

  be a graded, minimal free resolution of $\Gamma(X)$. By localizing $\mathcal{F}$ at the homogeneous maximal ideal M one still obtains a minimal free resolution. The assertion follows by Remarks A.40 and A.44.
- (iv) $\Leftrightarrow$ (i): it directly follows from the definition of dual module (see (A.45)) and the definition of dualizing sheaf.

□

Remark A.52. Observe that, if X is pG then $\mathcal{C}(X)$ is Gorenstein, hence X is Gorenstein, since it is an equidimensional hyperplane section of its affine cone.

Clearly, by Adjunction Formula and by Theorem A.31 and Proposition A.50, if X is pG, then each hyperplane section H of X not containing any component of X is pG. Conversely:

Proposition A.53. *Let X be an equidimensional, projective variety. If a equidimensional hyperplane section H of a projective variety X is pG, then X is pG too.*

Proof. It directly follows from Theorem A.48 and Proposition A.50. □

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